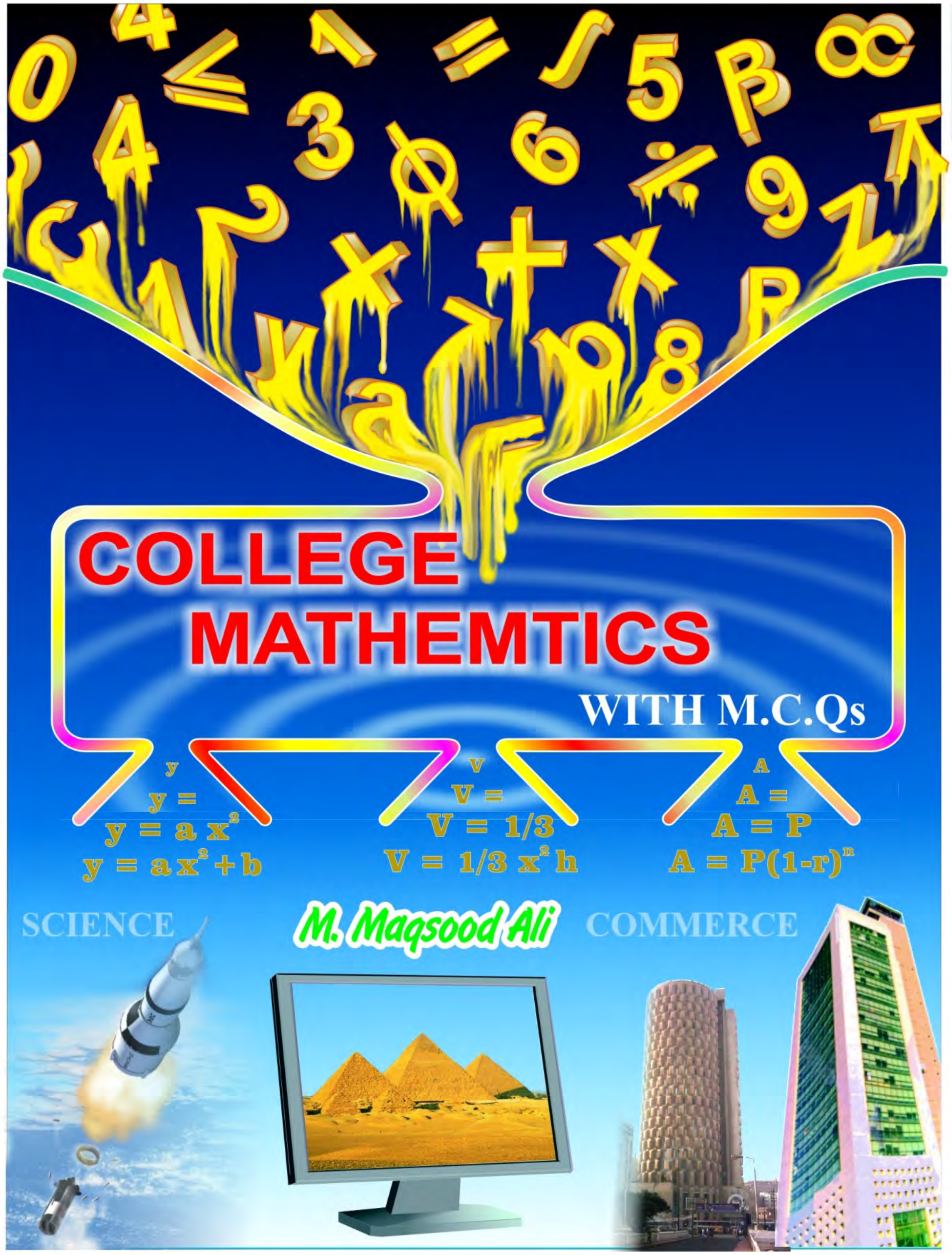

COLLEGE
MATHEMTICS
WITH M.C.Qs
y
y =
y = a x^2
y = ax^2+b
V
V =
V = 1/3
V = 1/3 x^{2}h
A
A =
A = P
A = P(1-r)^n
SCIENCE
M. Maqsood Ali
COMMERCE

SECTION C

ALGEBRA 1

The letters (x, y, z, a and b etc.) of the alphabet represent the numbers , the operations (+, −, x, ÷ etc. studied in arithmatics) and numbers are used to form equations or expressions or identities is related to a branch of mathematics is called Algebra.

For Example,

(1) $3x^2 - 9 = 0$ is an algebraic equation.

(2) $ax^2 + by^2$ is an algebraic expression.

(3) $(x + y)^2 = x^2 + 2xy + y^2$ is an algebraic identity.

Where x, y, z are variables and a, b are arbitrary constants.

VARIABLES

Variables are the numbers represented by letters x, y, z, t etc.

For example, the equation $x^2 - 4 = 0$ has two values of x that are 2 and −2.

Dependent and Independent Variables:

Consider the following equation of a straight line.

$$y = 2x + 1$$

By substituting the values of x, we get the values of y, as given below.

x	0	1	5	8
y	1	3	11	17

The value of y depends on the value of x, so y is dependent variable and x independent variable.

On a graph paper the values of independent variable are written on x-axis and the values of dependent variable on y-axis.

CONSTANTS

There are two types of constants in an equation.

(1) absolute constants (2) arbitrary constants

In the equation

$$y^2 = 2px^3 + 5q$$

2 and 5 are absolute constants and p and q are arbitrary constants because no values are specified for them.

Difference Between Variables and Arbitrary Constants:

$$y = mx + c \longrightarrow (1)$$

is an equation of a straight line.

Where m and c are arbitrary constants, because m has exactly one value for an equation and also c.

For example, when $m = 2$ and $c = 5$, equation (1) becomes

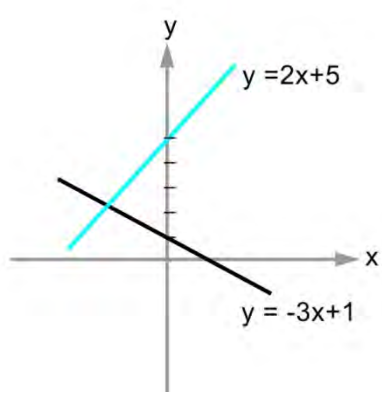
$$y = 2x + 5 \longrightarrow (2)$$

If we change the values of m and c, then equation (1) will be another equation of straight line different from equation (2).

For example, when $m = -3$ and $c = 1$, the equation (1) becomes

$$y = -3x + 1 \longrightarrow (3)$$

The graph of the lines $y = 2x + 5$ and $y = -3x + 1$, shown in the figure, are totally different, where (x, y) represents all the points lie on the lines.



EXPRESSION, EQUATION AND IDENTITY:

	Explanation	Example
Expression	An algebraic expression is a combination of numbers and variables by using the operations. Where, numbers: absolute constants and arbitarary constants. operations: addition, subtraction, multiplication and division etc.	(1) $x^2 - 6x + 8$ (2) $5x^3 + 3kx + 9$ where k is an arbitarary constant.
Equation	An algebraic equation is an equality of two algebraic expressions such that both sides are equal for some vlaues of the variables.	$x^2 + 6 = 5x$ Both sides are equal only for $x = 2$ and $x = 3$. But both sides are not equal for other values of $x \in \mathbb{R}$ and $x \in \mathbb{C}$
Identity	An algebraic identity is an equality of two algebraic expression such that both sides are equal for all values of the variables. It is possible that an identity may be undefined for some values of the variables.	(1) $(x+y)^2 = x^2 + 2xy + y^2$ Both sides are equal for all values of $x \in \mathbb{R}$. (2) $x + 1 = x^2 - 1/x - 1$ Both sides are equal for all $x \in \mathbb{R}$ but $x \neq 1$.

Chapter 7

EXPRESSION

A combination of symbols (representing operation numbers or other mathematical entities) and (addition, subtraction etc.) is said to expression. There is no sign of equality or inequality.

NUMERICAL EXPRESSION

A numerical expression includes at least one of the operations of addition, subtraction, multiplication or division and some numbers.

Following are the examples of numerical expressions.

$$100 + 2 - 3 \ , \ \frac{3}{2} \times \frac{5}{3} \ , \ \frac{2}{8} \div \frac{3}{6} + 9$$

ALGEBRAIC EXPRESSION

An expression forms from any combination of numbers and variables by using the operation of addition, subtraction, exponentition etc, is said to be algebraic expression. Following are the examples of algebraic expression.

$$x^2 \ , \ 8 \ , \ 6x + 9y \ , \ \frac{2x^2 + 3}{x + 1} + 3x^2 - 7 \ , \ \pi r^2 h \ , \ 3x^2 + 2xy - 9x$$

Term:

The parts connected by plus or minus sign in an algebraic expression is called a term.

For example, the algebraic expression $3x^2 + 9x - 7$ has three terms, that are $3x^2$, $9x$ and -7.

MULTINOMIALS

It is an algebraic expression consist of two or more terms.

(1) Monomials:

An algebraic expression consist of one term is called monomial.

The following are examples of monomials.

(a) $2x^2$ (b) $\frac{x}{2y}$ (c) $\sqrt{x}$ (4) $\frac{x}{y^2}$

(2) *Binomials:*

An algebraic expression with two terms is called binomial.

The following are the examples of binomials.

(1) $2x^2 + 3x$ (2) $\frac{2x}{y} + 3y^2$ (3) $\frac{x}{y^3} - 2x^{-2}$

(3) *Trinomials:*

An algebraic expression consist of three terms is called trinomi al.

The following are the examples of trinomials.

(1) $3\sqrt{x} + 2xy - 4$ (2) $\frac{3\sqrt{x}}{y^2} + 6y - 4x$ (3) $4x^3 + 3x^2 + 1$

POLYNOMIAL

An algebraic sum in which power of all variables are non-negative integers is called polynomial.

The following are the examples of polynomials.

(1) $3x^3 + 5x^2 + 2x + 1$ (degree = 3)
(2) $\sqrt{2}\,x^6 + 3x^2y^3 + 5xy$ (degree = 6)
(3) $3x^6 + 2xy + 5x^3y^2z^2$ (degree = 7)

In a term the sum of all exponents of the variables is called de gree of a term.

For example, the degree of the term $3x^2y^3z^4$ is 9. The degree of a polynomial is the highest degree of all terms with non-zero coefficient.

A polynomial must be a multinomial.

RULES OF SIMPLIFICATION

When an expression has more than one operations and brackets we simply it according to the given order.

Consider the following expression

$$3x + 14x \div (2 + 5) \times 6 - x$$

The simiplication step by step:

Step 1: Bracket (parentheses):

Simplify the expression first within the bracket.

$$3x + 14x \div 7 \times 6 - x$$

Step 2: Division and Multiplication:

Before addition or subtraction do division or multiplication left to right.

$$3x + 2x \times 6 - x$$
$$3x + 12x - x$$

Step 3: Addition and Subtraction:

At last do addition and subtraction left to right.

$$15x - x$$
$$14x$$

Example 1:

Simlify: $10x + 3x.4 + 16x \div (3-7) \times 20 - 2x$

Solution:-

$$10x + 3x.4 + 16x \div (3-7) \times 20 - 2x$$
$$= 10x + 3x.4 + 16x \div (-4) \times 20 - 2x$$
$$= 10x + 3x.4 - 4x \times 20 - 2x$$
$$= 10x + 12x - 80x - 2x$$
$$= 22x - 80x - 2x$$
$$= -58x - 2x$$
$$= -60x$$

BRACKETS

Symbol	Name
( )	parentheses
[ ]	brackets
{ }	braces
———	vinculum

Rules of Simplification of Brackets:

If there are more than one brackets in an expression, then we simplify the expression as given below.

$$
\begin{aligned}
&[5x - \{3x + (2x - \overline{x + 3})\}] \\
&= [5x - \{3x + (2x - x - 3)\}] \\
&= [5x - \{3x + (x - 3)\}] \\
&= [5x - \{3x + x - 3\}] \\
&= [5x - \{4x - 3\}] \\
&= [5x - 4x + 3] \\
&= x + 3
\end{aligned}
$$

L.C.M. AND L.C.D.:

See Arithmetic for L.C.M. and L.C.D.

Example 2: Find the L.C.M and H.C.F of the following:

$$5x^3y^2 \,,\; 25x^2y^2 \,,\; 10x^5y$$

Solution:-

$$\text{L.C.M} = 50x^5y^2$$
$$\text{H.C.F} = 5x^2y$$

Example 3: Find L.C.D. of the following:

$$\frac{3}{2x^2y^3} \,,\; \frac{12}{x^4y} \,,\; \frac{15}{4xy}$$

Solution:-

$$\text{L.C.D} = 4x^4y^3$$

FACTORIZATION

We will factorize the following three types of quadratic expression one by one.

(1) $x^2 - 3x$ (2) $x^2 - 25$ (3) $2x^2 + 13x + 18$

(1) $x^2 - 3x = x(x - 3)$

Remarks: No constant term.

(2) $x^2 - 25 = x^2 - 5^2$

$= (x - 5)(x + 5)$

Remarks: No term involving x.

(3) $2x^2 + 13x + 18$

Step 1: Multiply the coefficients of x^2 and constant term 18.

$$2 \times 18 = 36$$

Step 2: Break the middle term, such that the product of these terms is 36 and sum 13.

Step 3:

Trail	Product	Sum	Remarks
1, 36	36	37	no
2, 18	36	20	no
3, 12	36	15	no
4, 9	36	13	yes

Step 4:

$$\begin{aligned}2x^2 + 13x + 18 &= 2x^2 + 4x + 9x + 18\\ &= 2x(x + 2) + 9(x + 2)\\ &= (x + 2)(2x + 9)\end{aligned}$$

(33) $\dfrac{5}{2ab}, \dfrac{3}{4a^2b}, \dfrac{2}{ab^3}$ (34) $\dfrac{5x^2}{(x^2+y^2)}, \dfrac{15x^3}{(x^2+y^2)^3}$

(35) $\dfrac{2x}{(x+y)^3}, \dfrac{4x}{(x+y)^5}$

Factorize the following:

(36) $5a^2b + ab^3$ (37) $10m^4n^3 - 20m^2n$

(38) $3a^3b^3c^3 + 6a^2b^3c^2 - 9a^4bc^3$

(39) $6x^2y^3 + 3xy^4 - 9x^3y$

(40) $2(a+b)^2 + 5(a+b)$ (41) $14(x+y)^4 + 21(x+y)^3$

(42) $\dfrac{12x^2}{y^3} + \dfrac{6x^4}{y^5}$ (43) $\dfrac{5x^3}{(x+y)^2} + \dfrac{15x^4}{(x+y)^3}$

(44) $\dfrac{2(a+b)^3}{a^2b^3} + \dfrac{5(a+b)^4}{a^3b^2}$

Simplify the following:

(45) $6x^3 + 5x^8 - 3x^3 - 2x^8$ (46) $5x^2 - 10x^3 - 3x^2 + 10x^3$

(47) $6x^2 \times 2x^3 - 2x^5$ (48) $2a^5 \div 4a^3 \times 8a^2$

(49) $5x^3 \times 10x^5 \div 2x^4$ (50) $10x^3 + 5x^6 \div 2x^3$

(51) $5x^0 + 3x^2 \div 6x^2 - 6$ (52) $3x \times 5 \div 2x - 8$

(53) $2x^3 \times 5x^3 + 2x^3 \div 6x^3$ (54) $5x^2 \div 10x^2 \times 4x^3 - 6x^3$

(55) $10x^3 - 6x^3 \div 3x^2 \times 9x^2$ (56) $(3x^2 + 2x^3 \div x) \times 5x^3 + 8x^5$

(57) $(5x + 2x^2)^2 - 8x^2 \div 4x^{-5} \times 5x$

(58) $5x^2y^2 - \left\{2x^4 \div \left(\dfrac{x^2}{y^2} - \dfrac{5x^2}{y^2}\right)\right\}$

(59) $2x + [3x - \{5(2x - \overline{6x - 3})\}]$

(60) $2x^2 + \{3x^3 - (2x^4 \div 3x - 5x^3)\}$

Simplify the following:

(61) $\left(\dfrac{x}{y}\right)^3 \div \left(\dfrac{x^2}{y^3}\right)^o \times \left(\dfrac{x^2}{y^2}\right)^2$

(62) $\left(\dfrac{x^2y}{z^2}\right)^2 + \left(\dfrac{x^3y^2}{z^2}\right)^4 \div \left(\dfrac{x^4y^3}{z}\right)^4$

(63) $\left(\dfrac{x^2y^2}{z^4}\right)^3 \times \left(\dfrac{3xy^3}{z^2}\right)^2 - \left(\dfrac{x^4y^3}{z^4}\right)^4$

(64) $(2x^3y)^4 - (y^2)^3 \div (5x^{-6}y)^2$

(65) $(3x^{-2}z^3)^{-3} + (x^4z^3)^2 \times (4xz^{15/2})^{-2}$

(66) $x^{3/2} \div x^{5/2} \times x^2 + 5x$

(67) $x^{1/2} \div x^{-3/2} + x^{5/2} \times x^{-1/2}$

(68) $(x^{-1/2}y^{1/3})^3 \div (x^{-2/5}\,y)^5$

(69) $x^{-1/2} \div x^{-3/2} \times x^{2/3}$

(70) $x^o \div x^{-3/2} \div x^{5/2} \times x$

Simplify the following:

(71) $\dfrac{3}{6x^3} + \dfrac{5}{9x^2}$

(72) $\dfrac{5x}{2y^2} + \dfrac{6y}{10y^3}$

(73) $\dfrac{6x+5}{x+y} + 2$

(74) $\dfrac{5x+2y}{3} - \dfrac{2x-7y}{6}$

(75) $\dfrac{2(x+y)}{5} + \dfrac{x+7}{2} - y$

(76) $(5x+2)(3+2y) + \dfrac{2xy-7x}{4}$

(77) $\dfrac{3(5x+6y)}{5} - \dfrac{7(3x+5)}{10}$

(78) $\dfrac{5a}{(a+b)^2} - \dfrac{2}{(a+b)}$

(79) $\dfrac{5}{(x-y)^3} + \dfrac{6x}{(x-y)^2}$

(80) $\dfrac{2}{5xy^3} + \dfrac{7}{20x^2y} + 3$

EXERCISE C-2

Factorize the following expressions:

(1) $2x + xy^2$

(2) $3xy^2 + 15x^2y + 9x^2y^2$

(3) $3ab + 9a^2b^2$

(4) $15a^2b + 10a^3c$

(5) $2x^2 + 6x^4$

Factorize the following expressions:

(6) $a + b + (a + b)^2$

(7) $2(a + b)^2 + a + b$

(8) $5(a - b)^2 - 2a + 2b$

(9) $5x - 5y - 2ax + 2ay$

(10) $3x^2 - 9xy + xy - 3y^2$

Factorize the following expressions:

(11) $x^2 - 4$

(12) $9a^2 - b^2$

(13) $4x^2 - 9y^2$

(14) $16m^2 - 25n^2$

(15) $3x^2 - 5y^2$

(16) $x^2 - y^2 + x + y$

(17) $a^2 - b^2 - a + b$

(18) $4a^2 - b^2 + 2a - b$

(19) $a + 5b + a^2 - 25b^2$

Factorize the following expression:

(20) $x^2 + 5x + 6$

(21) $x^2 - 7x + 10$

(22) $x^2 + 4x + 4$

(23) $x^2 - 6x + 9$

(24) $x^2 + 2x - 15$

(25) $x^2 - 3x - 18$

(26) $2x^2 + 7x + 6$

(27) $3x^2 - 17x + 10$

M.C.Q's C-2

(1) L.C.M and H.C.F of $6x^3y^2$, $10xy$ are respectively ______.

(a) $60x^4y^3$, $15xy$ (b) $30x^3y^3$, $2xy$

(c) $30x^3y^2$, $2xy$ (d) $30x^3y^2$, $2x^3y^2$

(2) L.C.M and H.C.F of $15x^2y^3z^0$, $6x^4y^2z^2$ and $10x^3y^5z$ are respectively _____.

(a) $900x^9y^{10}z^3$, $2x^2y^2z$ (b) $300x^4y^5z^2$, $2x^2y^2$

(c) $15x^2y^2z^0$, $60x^4y^5z^2$ (d) $30x^4y^5z^2$, x^2y^2

(3) What is the L.C.M of $\frac{8}{x^3}, \frac{6}{x^5}$?

(a) x^5 (b) x^8 (c) $\frac{48}{x^8}$ (d) $\frac{24}{x^5}$

(4) What is the L.C.M of $\frac{x^8}{12}, \frac{x^6}{18}$?

(a) $\frac{x^8}{36}$ (b) $\frac{x^{14}}{216}$ (c) 36 (d) 216

(5) What is the L.C.M of $\frac{16}{(x^2+y^2)}, \frac{12}{(x+y)^2}$?

(a) $\frac{48}{(x+y)^2}$ (b) $(x^2+y^2)(x+y)^2$

(c) $\frac{48}{(x^2+y^2)(x+y)^2}$ (d) $\frac{48}{x^2+y^2}$

(6) What is the L.C.D of $\frac{3}{x^3}, \frac{15}{x^4}, \frac{20}{x^6}$?

(a) 60 (b) x^6 (c) $\frac{60}{x^6}$ (d) $\frac{900}{x^{13}}$

(7) What is the L.C.D of $\frac{1}{6x^3}, \frac{1}{9x^5}$?

(a) $\frac{1}{54x^8}$ (b) $54x^8$ (c) $\frac{1}{18x^5}$ (d) $18x^5$

(8) What is the L.C.D of $\frac{12}{ax+a}, \frac{16}{x}$?

(a) $\frac{48}{ax+a}$ (b) $\frac{48}{a(x+1)x}$ (c) $a(x+1)$ (d) $ax\,(x+1)$

(a) 29 (b) 25 (c) 9 (d) 33

(20) If $3a + 2b = 4$ and $6ab = 24$, then $9a^2 + 4b^2 - 2ab = ?$

(a) 16 (b) -40 (c) -52 (d) None

(21) If $3a - 5b = 4$ and $9a^2 = 20 + 25b^2$ then $3a + 5b = ?$

(a) 5 (b) 400 (c) 100 (d) None

(22) If $a + 2b = 5$ and $5a - c = 6$ then $5a^2 + 10ab - ac - 2bc = ?$

(a) 1 (b) 11 (c) 30 (d) None

(23) If $a + 3b = 7$ and $6ab - c^2 = 2$ then $a^2 + 9b^2 + c^2 = ?$

(a) 40 (b) 27 (c) 45 (d) None

(24) If $a - b = 6$ and $c^2 + ab = 5$, then $a^2 + b^2 + 2c^2 = ?$

(a) 41 (b) 36 (c) 46 (d) None

(25) If $3x + 4y = 9$ and $4y - 3x = 3$, then $9x^2 - 16y^2 = ?$

(a) 27 (b) 243 (c) -27 (d) None

(26) $5^{2x+1} = 1$, then $x = ?$

(a) $5^{-1/2}$ (b) $-1/2$ (c) $\sqrt{5}$ (d) 0

(27) $125^{2x} = 5^2$ then $x = ?$

(a) $2/5$ (b) 1 (c) $2/3$ (d) $1/3$

(28) $54^x = 27 \times 6^x$, then $x = ?$

(a) $3/2$ (b) $x = 3$ (c) $5/2$ (d) 2

(29) $(0.008)^x = 4^{2x+1}$ then $x = ?$

(a) $-7/3$ (b) $3/2$ (c) $5/3$ (d) $-2/3$

(30) $(0.02)^x = 8 \times 5^x$ then $x = ?$

(a) 2 (b) $-3/2$ (c) $-1/2$ (d) -2

Chapter 8

EQUATIONS

An equation is a equality of two algebraic expressions. An equation consists of one or more than one variable.

Following are the examples of equations.

(1) $5x^3 + 2x + 5 = 0$ (one variable, x)

(2) $3x^2y + 5x = 9y^2 + c$ (two variables x and y)

(3) $5ax^3 + 9bxyz = 6$ (three variables x, y and z)

(4) $ax^m + 5x^2 + 1 = 0$ (one variable x)

Note: x^m : x is said to be base and m exponent or power or index.

POLYNOMIAL EQUATION

An equation in which both sides of the equality are polynomials is called polynomial equation.

An equation is called is standard form if all the terms are written on left side and zero on right side.

$$ax^m + bx^{m-1} + \ldots + cx + d = 0$$

If the equation is in standard form the degree of the equation is the degree of the polynomial of left side.

ROOTS OF A POLYNOMIAL EQUATION:

The number of roots of a polynomial equation depend on the degree of the polynomial equation. Such that

(1) $x^3 - 3x^2 + 5x + 7 = 0$ (degree 3, roots are not more than 3)

(2) $7x^5 + 2x^4 - 2x^2 + 7x - 3 = 0$ (degree 5, roots are not more than 5)

(3) $3x^7 - 2x^6 + 2x^3 + 2x - 9 = 0$ (degree 7, roots are not more than 7)

Note: A polynomial equation can not have roots more than its degree.

NUMBER OF POSITIVE AND NEGATIVE ROOTS

French Mathematician Rene Descartes (seventeenth century) discovered a rule for a polynomial equation $P(x) = 0$ with real coefficients and arranged in descending power of x, that

(1) The number of positive real roots of a polynomial equation $P(x) = 0$ is either equal to the number of variation of sign for $P(x)$, or less than that number by an even integer.

(2) The number of negative real roots of a polynomial equation $P(x) = 0$ is either equal to the number of variation of sign for $P(-x)$, or less than that number by an even integer.

Note: The terms with zero coefficients (missing terms) must be ignored when counting the total number of variations of sign.

Example 1: Determine the possible number of positive and negative roots of $5x^7 - 3x^5 + 2x^4 + x^2 - x - 9 = 0$.

Solution:-

$$P(x) = 5x^7 \overset{1}{-} 3x^5 \overset{2}{+} 2x^4 + x^2 \overset{3}{-} x - 9 = 0$$

$P(x)$ has three variations of sign.

Therefore, equation has 3 or 1 positive roots.

Since,

$$P(-x) = 5(-x)^7 - 3(-x)^5 + 2(-x)^4 + (-x)^2 - (-x) - 9 = 0$$

$$P(-x) = -5x^7 \overset{1}{+} 3x^5 + 2x^4 + x^2 + x \overset{2}{-} 9 = 0$$

$P(-x)$ has two variation of sign.

Therefore, equation has 2 or 0 negative roots.

Example 2: Find the possible roots of the following equation.

$$2x^3 + x^2 - 13x + 6 = 0$$

Solution:-

$$2x^3 + x^2 - 13x + 6 = 0$$

$a_o = 6 \quad \Rightarrow p = \pm 1, \pm 2, \pm 3, \pm 6$ {possible integer roots}

$a_3 = 2 \quad \Rightarrow q = \pm 1, \pm 2$

Possible rational roots are

$$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

where $\pm\frac{1}{2}, \pm\frac{3}{2}$ are possible fraction roots

Since,

$$P(x) = 2x^3 + x^2 - 13x + 6$$

Two variation s of sign. Therefore, number of positive real roots are 2 or 0.

$$P(-x) = -2x^3 + x^2 + 13x + 6$$

One variation of sign. Therefore, number of negative real roots are 1.

Test for Roots:-

$P(0) = 6 \neq 0$

$P(1) = 2(1)^3 + 1^2 - 13(1) + 6 = -4 \neq 0$

{one root lies between 0 and 1}

$P(2) = 2(2)^3 + 2^2 - 13(2) + 6 = 0$ {2 is a root}

Since this equation has at most two positive roots.

Now we test for negative roots.

$P(-1) = 2(-1)^3 + (-1)^2 - 13(-1) + 6 = 18 \neq 0$

$P(-2) = 2(-2)^3 + (-2)^2 - 13(-2) + 6 = 20 \neq 0$

$P(-3) = 2(-3)^3 + (-3)^2 - 13(-3) + 6 = 0$ {-3 is a root}

One root lies between 0 and 1, trial $1/2$.

$$P\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 13\left(\frac{1}{2}\right) + 6 = 0 \qquad \{1/2 \text{ is a root}\}$$

So that -3, $1/2$ and 2 are the roots of the equation.

REAL ROOTS OF AN EQUATION BY THE GRAPH

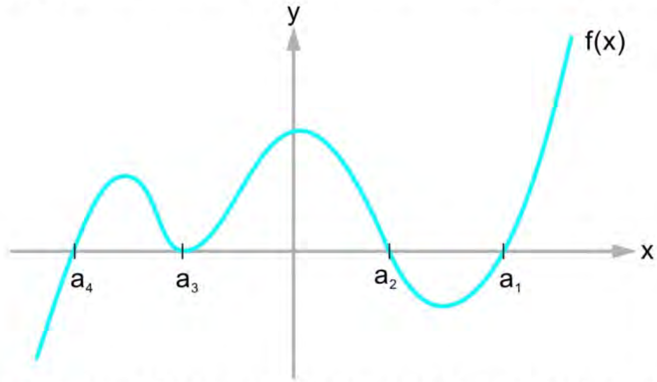
Consider the following equation in x.

$$ax^m + bx^n + \ldots + cx + d = 0$$

Suppose that

$$f(x) = ax^m + bx^n + \ldots + cx + d = 0$$

The graph of f in xy-plane is shown given below.



As we know that, p is the root of the equation if $f(p) = 0$.

Since the value of f(x) is zero on x-axis, hence the real roots of the equation are that where the curve intersects or touches x-axis. So that a_1, a_2, a_3 and a_4 are real roots of the above equation.

EXERCISE C-3

Write down the possible number of positive and negative real roots. Also write down all expected roots of the equation.

(1) $x^3 + 4x^2 + x - 6 = 0$

(2) $x^4 + 4x^3 - x^2 - 16x - 12 = 0$

(3) $x^3 - 4x^2 - 25x + 28 = 0$

(4) $x^3 + 5x - 4x^2 - 20 = 0$

(5) $x^4 + 9x^3 + 13x^2 - 9x - 14 = 0$

(6) $x^4 + 10x + 2x^3 + 2x^2 - 15 = 0$

(7) $5x^3 - 7x^2 - 28x + 12 = 0$

(8) $3x^3 + x^2 - 8x + 4 = 0$

(9) $5x^3 + 35x + 26x^2 + 6 = 0$

(10) $2x^3 - 9x^2 + 13x - 6 = 0$

M.C.Q's C-3

(1) How many positive real roots have the equation
$x^4 - 9x + 9x^3 + 13x^2 - 14 = 0$?
(a) 1 (b) 3 (c) 5 (d) 2

(2) How many negative real roots have the equation
$x^4 + 9x^3 + 13x^2 - 9x - 14 = 0$?
(a) 5 (b) 2 (c) 3 (d) 0

(3) How many negative roots have the equation
$x^4 + 2x^3 + 2x^2 + 10x - 15 = 0$?
(a) 4 (b) 1 (c) 2 (d) 5

(4) How many positive real roots the following equation has
$x^4 + 2x^2 + 10x + 2x^3 - 15 = 0$?
(a) 4 (b) 0 (c) 3 (d) 1

(5) How many negative roots have the equation
$x^5 - 5x^4 + 2x^3 + 4x^2 - 8x + 96 = 0$?
(a) 4 (b) 1 (c) 5 (d) 2

(6) How many positive real roots have the equation
$x^5 - 5x^4 + 2x^3 + 4x^2 - 8x + 96 = 0$?
(a) 1 (b) 5 (c) 3 (d) 2

(7) How many positive real roots have the equation
$x^4 + 11x^3 + 41x^2 + 61x + 30 = 0$?
(a) 3 (b) 1 (c) 0 (d) 4

(8) How many negative real roots have the equation
$x^4 + 11x^3 + 41x^2 + 61x + 30 = 0$?
(a) 5 (b) 1 (c) 3 (d) 4

(9) How many negative real roots have the equation
$x^3 + 9x - 6x^2 - 54 = 0$?
(a) 0 (b) 3 (c) 2 (d) 4

(10) What is a root of the equation $x^3 + 5x^2 - 17x - 21 = 0$?
(a) 2 (b) 5 (c) 11 (d) -7

(11) What is a root of the equation $x^4 - 7x^3 + 16x^2 - 28x + 48 = 0$?
(a) 5 (b) 4 (c) -7 (d) 9

(12) What is a root of the equation $x^4 - 6x^3 + 17x^2 + 24x - 24 = 0$?
(a) 7 (b) -5 (c) 3 (d) -13

(13) What is a root of the equation $x^3 - 16x^2 + 71x - 56 = 0$?
(a) 8 (b) -7 (c) -9 (d) 17

(14) What is a root of the equation $x^4 + 11x^3 + 41x^2 + 61x + 30 = 0$?
(a) -7 (b) 7 (c) 5 (d) -5

(15) What is a root of the equation $x^3 - 12x^2 + 41x - 42 = 0$?
(a) -9 (b) -7 (c) 5 (d) 7

(16) What is a root of the equation $2x^3 + 13x^2 + 17x - 12 = 0$?
(a) $5/3$ (b) $1/2$ (c) 5 (d) -7

(17) What is a root of the equation $3x^3 + x^2 - 8x + 4 = 0$?
(a) $2/3$ (b) $5/3$ (c) $3/2$ (d) -7

(18) What is a root of the equation $3x^3 + x^2 - 8x + 4 = 0$?
(a) $5/3$ (b) $2/3$ (c) $8/3$ (d) $-7/3$

(19) What is a root of the equation $5x^3 + 26x^2 + 35x + 6 = 0$?
(a) $3/5$ (b) $2/5$ (c) $-1/5$ (d) $-4/5$

(20) What is a root of the equation $2x^3 - 9x^2 + 13x - 6 = 0$?
(a) $-3/2$ (b) $-1/2$ (c) $5/2$ (d) $3/2$

Example 1: Three years ago the age of Sarim was double the age of Asif. At present Sarim is 5 years older than Asif. Find the age of sarim now.

Solution:-

Sarim's age = x

Asif's age = x − 5

Three years ago

$$(x-3) = 2(x-5-3)$$
$$x = 13$$

Sarim's age is 13 years.

Example 2: Asif, Wasim and Ali share Rs.569. Asif share is 5 less than double the share of Ali and Ali's share is three fifth the share of Wasim. Find the shares of Asif, Wasim and Ali.

Solution:-

Wasim' s share = x

$$\text{Ali's share} = \frac{3}{5}x$$

$$\text{Asif's share} = 2\left(\frac{3x}{5}\right) - 5 = \frac{6x-25}{5}$$

So that,

$$x + \frac{3x}{5} + \frac{6x-25}{5} = 569$$
$$x = 205$$

∴ Rs.205, Rs.241 and Rs.123 are the shares of Wasim, Asif and Ali respectively.

Example 3: A cyclist leaves his house at 10 : 45 a.m. and reaches to a shop at a distance 2 km from his house at an average speed 4 km/h and than walks 2 hours 15 minutes to reach his office. At what p.m. he will be at his office.

Solution:-

$t_1 = \dfrac{x}{v} = \dfrac{2}{4} = \dfrac{1}{2} h = 30$ minutes

Time to reach office = 10:45 + 00.30 + 02:15

= 13:30

The cyclist will reach office at 1:30 p.m.

M.C.Q'S C-4

(1) 5 more than twice a number is 49. What is the number?

(a) 103 (b) 29.5 (c) 22 (d) 27

(2) The product of two numbers x and five minus the quotient of x an d 6 is zero. What are the numbers?

(a) $0, -5/6$ (b) 0, 30 (c) −5, 1 (d) 2, 1

(3) 3 less than five times a number. What is the algebraic expressio n?

(a) $5x - 3$ (b) $5x + 3$ (c) $3 - 5x$ (d) 3 @5x

(4) Nine times a number decreased by 7 is 5. What is the algebraic expression?

(a) $7 - 9x = 5$ (b) $9(x - 7) = 5$

(c) $9x - 5 = 7$ (d) $9x - 7 = 5$

(5) Ali is four years older than Yasir and Yasir is two years younger than Kashif. What is the age of Ali if the sum of their ages is 18.

(a) 6 (b) 9 (c) 8 (d) 5

(6) Two years ago Khalid's age was three times the age of Hamid. Hamid is three years younger than Talib now. What is the age of Khalid if Talib is 11 years old now.

(a) 20 (b) 22 (c) 30 (d) 18

(7) Ali, Babar and Khalid share Rs.569. Ali's share is 5 less than d ouble the share of Khalid and Khalid's share is three fifth the share of Babar. What is the share of Ali?

(a) 123 (b) 241 (c) 205 (d) 307

(8) Asif, Bashir and Salman share Rs.900. Asif's share is two third the share of Bashire and Salman's share is twice the sum of the share of both. What is the share of Asif ?

(a) 180 (b) 600 (c) 120 (d) 150

(18) The salary of a sales man is \$200 plus commission 40 cent per bundle after selling 100 bundles. If he sales 250 bundles, how much does he receive this month?

(a) \$300 (b) \$260 (c) \$250.10 (d) None

(19) The price of petrol is Rs.36 per litre. Ali's car travels 12km per litre. How many kilometers he travels if he has petrol in the car of Rs.x.

(a) 3x (b) $^3/_x$ (c) $^x/_3$ (d) None

(20) Ali has a car. The car is gone 10km per litre and Ali has petrol of Rs.600 in his car and travel x km. What is the price in rupees of the petrol per litre.

(a) $^{6000}/_x$ (b) $^{60}/_x$ (c) $^x/_{60}$ (d) $^x/_6$

[For more problems see also topics "system of two equations" and "rate"].

Chapter 8B

QUADRATIC EQUATIONS

The equation in the form

$$ax^2 + bx + c = 0 \;,\; a \neq 0$$

is a second degree polynomial equation or quadratic equation in x.

SOLUTION OF A QUADRATIC EQUATION

We discuss three methods to determine the roots of a quadratic equation.

(1) Factorization (2) Completing the square

(3) Quadratic Formula

(1) Determining Roots by Factoring:

(i) $x^2 + 5x = 0$ no constant term.

Factoring the equation

$$x(x + 5) = 0$$

Either $x = 0$ or $x + 5 = 0$

=> $x = 0$ or $x = -5$

Therefore, 0 and -5 are the roots of the equation.

(ii) $2x^2 + 5x + 3 = 0$

Break 5x into two terms such that the sum of the terms is 5x and product $6x^2$.

Trial	Product	Sum	Remarks
1 , 6	6	7	no
2 , 3	6	5	yes

$$2x^2 + 2x + 3x + 3 = 0$$
$$2x(x + 1) + 3(x + 1) = 0$$
$$(x + 1)(2x + 3) = 0$$

Either $x + 1 = 0$ or $2x + 3 = 0$

=> $x = -1$ or $x = -3/2$

Therefore, the roots of the equation are -1 and $-3/2$.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3: Adding both sides $\left(\frac{\text{coefficient of } x}{2}\right)^2$ that is $\left(\frac{b}{2a}\right)^2$.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$

Step 4:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2\ 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is called quadratic formula.

(3) Determining Roots Using Quadratic Formula:

We solve the following quadratic equation using quadratic formula.

$$x^2 - 5x + 6 = 0$$

The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By substituting $a = 1$, $b = -5$ and $c = 6$, we have

$$x = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm 1}{2}$$

Either $x = \frac{5+1}{2}$ or $x = \frac{5-1}{2}$

$x = 3$ or $x = 2$

EXERCISE C-4

Factorize the following equations and hence find the roots of the equations:

(1) $x^2 - 2x + 1 = 0$

(2) $x^2 + 3x - 10 = 0$

(3) $t^2 - 5t + 6 = 0$

(4) $3y^2 + 13y + 14 = 0$

(5) $5u^2 - 7u - 6 = 0$

Using completing the square method, find the roots of the following equations:

(6) $x^2 + 4x = 5$

(7) $x^2 - 6x = 16$

(8) $2x^2 + 6x - 8 = 0$

(9) $3x^2 = 5x + 1$

(10) $8x - 3x^2 = 3$

Using quadratic formula, find the roots of the following equations:

(11) $x^2 + 6x + 9 = 0$

(12) $3x^2 - 5x + 25 = 0$

(13) $2x^2 - 6x - 20 = 0$

(14) $x^2 - 10x - 30 = 0$

(15) $x^2 + 5x + 25 = 0$

NATURE OF THE ROOTS OF A QUADRATIC EQUATION

A quadratic equation has at most two roots. These roots may be real distinct, real equal, rational distinct or complex. We can determine the nature of the roots without solving the equation, using discriminant. The expression $b^2 - 4ac$ appearing under the radical in quadratic formula is called discriminant. To determine the nature of the roots of the quadratic equation $ax^2 + bx + c = 0$, we find the value of $D = b^2 - 4ac$. There are four cases.

(1) $D > 0 \Leftrightarrow$ the roots are real and distinct.

(2) $D = 0 \Leftrightarrow$ the roots are real and equal.

(3) $D < 0 \Leftrightarrow$ the roots are complex and distinct.

(4) D is perfect square $\Leftrightarrow$ the rotos are rational and distinct.

According to (1) and (2) we can say that

$$D \geq 0 \Leftrightarrow \text{the roots are real.}$$

Example 1: Determine the nature of the roots of the following equations.

(1) $x^2 - 5x + 6 = 0$ (2) $x^2 - 4x + 4 = 0$

(3) $x^2 + 3x + 5 = 0$ (4) $x^2 + 5x + 4 = 0$

Solution:-

Since, $D = b^2 - 4ac$

(1)

$$D = (-5)^2 - 4(1)(6) = 1 > 0$$

The roots of the equation are real and distinct.

(2)

$$D = (-4)^2 - 4(1)(4) = 0$$

The roots of the equation are real and equal.

(3)

$$D = (3)^2 - 4(1)(5) = -11 < 0$$

The roots are complex and distinct.

(4)

$$D = (5)^2 - 4(1)(4) = 9$$

9 is perfect square of 3.

So that the roots are rational and distinct.

APPLICATION OF DISCRIMINANT TO COORDINATES GEOMETRY:

(i) *Curve and x-axis:*

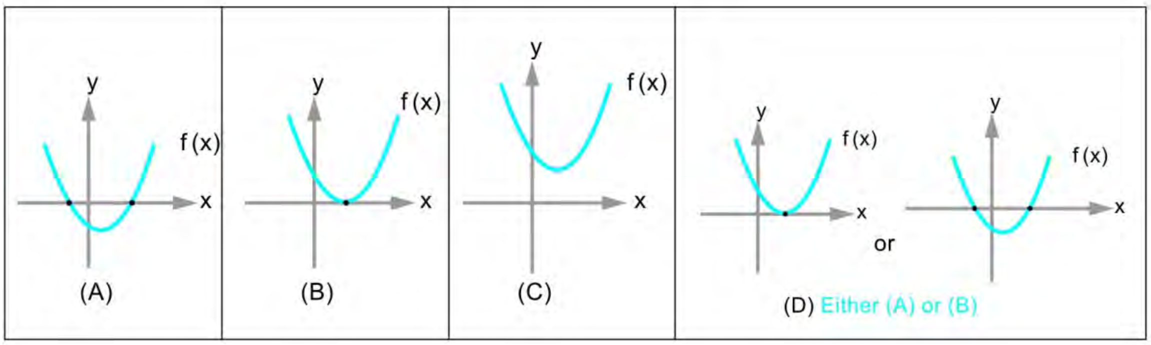
The euqation of x-axis:

$$y = 0 \longrightarrow (1)$$

The equation of curve:

$$f(x) = y = ax^2 + bx + c = 0 \ , \ a > 0 \longrightarrow (2)$$

There are following three types of graphs of the curve in xy-plane.



To find the abscissas of point of intersection of the curve and x-axis, substituating $y = 0$ from equation (1) in equation (2), we get

$$ax^2 + bx + c = 0 \longrightarrow (3)$$

The discriminant of equation (3) tell us the behaviour of the curve with the x-axis.

(a) The curve interest x-axis at two points $\Leftrightarrow$ $D > 0$
it means roots of the equation (3) are real and distinct.

(b) The curve touches x-axis $\Leftrightarrow$ $D = 0$
it means roots of the equation (3) are real and equal.

(c) The curve does not touch the x-axis $\Leftrightarrow$ $D < 0$
it means roots of the equation (3) are complex and distinct.

(d) The curve meets the x-axis $\Leftrightarrow$ $D \geq 0$
it means roots of the equation (3) are real distinct or equal.

(ii) Curve and Straight Line:

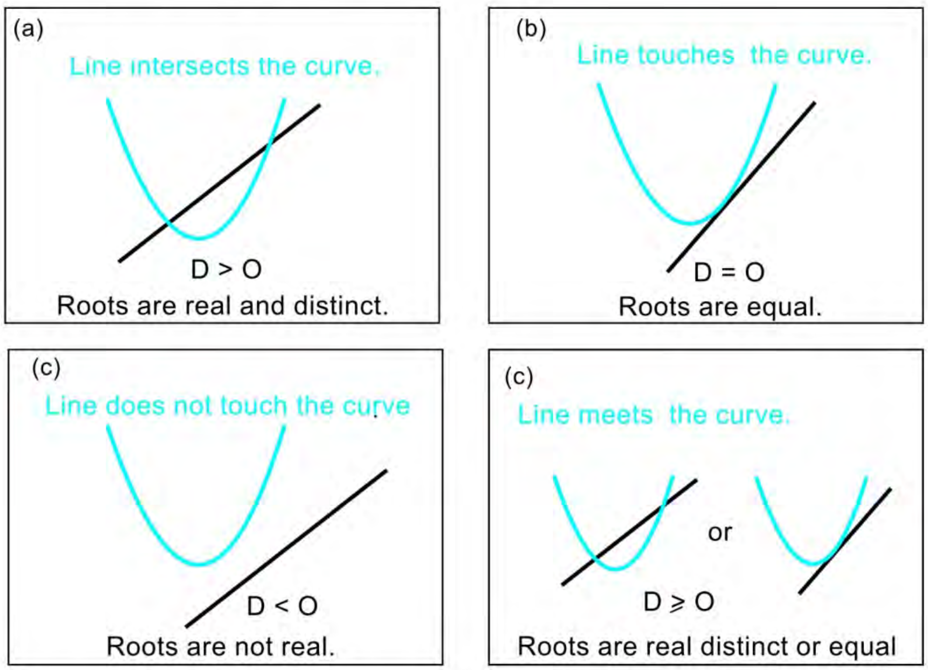
The equation of straight line:

$$y = mx + c' \longrightarrow (1)$$

The equation of curve:

$$y = ax^2 + bx + c \longrightarrow (2)$$

There are three possibilities (A), (B) and (C) as shown in the figure to draw the curve and straight line.



To find the abscissas of points of intersection of the curve and straight line, substituting y = 0 from equation (1) in equaiton (2), we get

$$ax^2 + bx + c = mx + c'$$

$$ax^2 + (b - m)x + (c - c') = 0 \longrightarrow (3)$$

$$A = a \ , \ B = b - m \ , \ C = c - c'$$

$$D = B^2 - 4AC$$

(a) Line intersects the curve $\Leftrightarrow$ $D > 0$

(b) Line touches the curve $\Leftrightarrow$ $D = 0$

(c) Line does not touch the curve $\Leftrightarrow$ $D < 0$

(d) Line meets the curve $\Leftrightarrow$ $D \geq 0$

M.C.Q's C-5

(1) What is the nature of the roots of the equation $x^2 - 5x + 6 = 0$?
(a) irrational (b) real equal
(c) not real (d) rational

(2) Let D be the discriminant of the quadratic equation $ax^2 + bx + c = 0$. The roots of the equation are real if
(a) $D = 0$ (b) $D \geq 0$ (c) $D < 0$ (d) $D > 0$

(3) The roots of a quadratic equation are not real if
(a) $D = 0$ (b) $D < 0$ (c) $D \geq 0$ (d) $D > 0$

(4) The roots of a quadratic equation are real and distinct if
(a) $D > 0$ (b) $D < 0$ (c) $D = 0$ (d) None

(5) The curve $y = 2x^2 - 8x + b$ touches x-axis, if
(a) $b = 8$ (b) $b \leq 8$ (c) $b > 8$ (d) $b \geq 0$

(6) The roots of the equation $2x^2 - mx - 5 = 0$ are _______, where m is a real number.
(a) equal (b) real (c) complex (d) None

(7) The roots of the equation $x^2 - 6x + m^2 = 2mk$ are equal if $m^2 - 2mk = ?$
(a) 5 (b) 0 (c) $-k^2$ (d) 9

(8) What is the least integer added or subtracted in the equation $x^2 + 9x = 0$, that the roots of the equation must be complex.
(a) 20 (b) 21 (c) 3 (d) 1

(9) What is the least integer should be added that the roots of the equation $x^2 - 6x = 0$ must be not real.
(a) 9 (b) 8 (c) 3^2 (d) 10

(10) What is the greatest integer should be added that the roots, except 0 and -8, of the equation $x^2 + 8x = 0$ must be real.
(a) 16 (b) 15 (c) 17 (d) 20

(11) What is the greatest integer should be added that the roots, except 0 and 7, of the equaiton $x^2 - 7x = 0$ must be real and distinct.
(a) 13 (b) -13 (c) 12 (d) 11

(12) What is the real number should be added in the equaiton $2x^2 - 5x = 0$ that the roots of the equation must be equal.
(a) 25 (b) $25/4$ (c) $49/16$ (d) $25/16$

(13) The equation $x^2 - 10x - c = 0$ have real roots, c is an integer, the least value of c is ________.

(a) 0 (b) -26 (c) -25 (d) -24

(14) The equation $2x^2 - 9x - m = 0$ have real roots, where m is an integer. The least value of m is ________.

(a) -11 (b) -10 (c) -9 (d) -10.125

(15) The equation $x^2 - kx + 25 = 0$ have real roots. The range of the values of k is ________.

(a) $k \geq \pm 10$ (b) $k > \pm 10$

(c) $k \leq -10, k \geq 10$ (d) $k > 10$

(16) The equation $px^2 - 10x + p = 0$ have real and distinct roots. The range of the values of p is ________.

(a) $p < \pm 5$ (b) $p < 5$

(c) $-5 \leq p \leq 5$ (d) $-5 < p < 5$

(17) The line $y = 2$ is tangent to the curve $y = x^2 - 8x + p$, where p is an integer. The value of p is ________.

(a) 16 (b) 18 (c) 12 (d) 49

(18) The line $y = 1$ intersects the curve $y = x^2 - 4x + p + 1$. The greatest value of p is ______, where p is an integer.

(a) 4 (b) 5 (c) 0 (d) 3

(19) The line $y = -3$ does not meet the curve $y = x^2 + 6x + p - 3$, where p is an integer. The least value of p is ________.

(a) 10 (b) 9 (c) 8 (d) 3

(20) The line $y = 5$ does not intersect the curve $y = x^2 - 4x - k + 5$, where k is an integer. The greatest value of k is ________.

(a) -3 (b) -5 (c) -4 (d) 4

SUM AND PRODUCT OF THE ROOTS

$ax^2 + bx + c = 0$ is a quadratic equation. Let α and β be the roots of this equation.

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum of the Roots:

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{b}{a}$$

Product of the Roots:

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{c}{a}$$

Example 2: Find the sum and product of the roots of the following equations:

$$x^2 - 5x + 6 = 0$$

(1) without using formula.

(2) using Formula.

Solution:-

$$x^2 - 5x + 6 = 0$$

$a = 1$, $b = -5$, $c = 6$

(1) The roots of the equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$$= 2, 3$$

Let $\alpha = 2$ and $\beta = 3$.

EXERCISE C-6

Find the sum and product of the roots of the following equations:

(a) without using formula. (b) using formula

(1) $x^2 + 7x + 10 = 0$ (2) $3x^2 - 7x - 6 = 0$

(3) $x^2 - 9 = 0$ (4) $2x^2 + 3x + 2 = 0$

Find the equation whose roots are given below.

(5) 3 and 6 (6) $-3/2$ and $1/4$

(7) $2i$ and $-2i$ (8) $3 + \sqrt{5}$ and $3 - \sqrt{5}$

(9) Find the equation whose roots are tripled the roots of the equation $x^2 - 6x - 16 = 0$.

(10) Find the equation whose roots are three times plus 5 the roots of the equation $x^2 - 7x + 10 = 0$.

(11) Find the equation whose roots are four less than the roots of the equation $5x^2 + 13x - 6 = 0$.

(12) Find the equation whose one root is 12 and other root is five times minus four the first root.

(13) α and β are the roots of the equation $ax^2 + bx + c = 0$; $a \neq 0$. Find the equation whose roots are $\alpha^2 + 1$ and $\beta^2 + 1$.

(14) α and β are the roots of the equation $x^2 + bx + c = 0$. Find the equation whose roots are $2\alpha^3 - 3$ and $2\beta^3 - 3$.

M.C.Q's C-6

(1) What is the sum of two roots of the equation $5x^2 - 8x - 4 = 0$?

(a) $\frac{1}{2}$ (b) $\frac{5}{8}$ (c) $\frac{8}{5}$ (d) $\frac{4}{5}$

(2) What is the product of two roots of the equation $2y^3 + 11y^2 + 12y = 0$?

(a) None (b) 6 (c) -6 (d) $-11/2$

(3) What is the sum of the roots of the equation $2y^3 + 4y^2 - 16y = 0$.

(a) None (b) -8 (c) 2 (d) -2

(4) What is the product of two roots of the equation $3y^3 - 7y^2 - 6y = 0$?

(a) -2 (b) $7/3$ (c) 0 (d) None

(5) What is the product of two roots of the equation $2y^4 + 29y^2 + 50 = 0$?

(a) 50 (b) -5 (c) 25 (d) $-29/2$

(6) What is the product of two roots of the equation $y^4 - 5y^2 + 9 = 0$?

(a) -3 (b) 9 (c) 5 (d) None

(7) The product of one root to the other root of the equation $2x^2 - 5kx + k = 2$ is 8. What is the value of k?

(a) 12 (b) 18 (c) 16 (d) 20

(8) The square of the sum of the roots of the equation $3x^2 - 6kx + 10 = 0$ is 36. What is the value of k?

(a) -3 (b) 9 (c) $\sqrt{18}$ (d) -6

(9) The sum of the roots of the equation $x^2 - 2kx + 12 = 0$ is equal to 4 times the product of the roots. What is the value of k?

(a) 12 (b) 24 (c) 48 (d) 6

(10) The product of the roots of the equation $2x^2 + 6x + k = 1$ is equal to the square of the sum of the roots. What is k?

(a) -6 (b) 7 (c) 19 (d) 18

(11) The product of the roots of the equation $3x^2 - 12x + p = 6$ is equal to the square root of sum of the roots. What is the value of p?

(a) 12 (b) 6 (c) 18 (d) 54

(12) α and β are the roots of the equation $5x^2 + 3kx + 20 = 5x$. What is the value of k if square root of sum of the roots is equal to the product of the roots?

(a) 15 (b) -5 (c) -25 (d) None

(13) What is the sum of the square of the roots of the equation

$3x^4 - 12x^2 + 5 = 0$?

(a) -16 (b) 2 (c) 16 (d) 4

(14) What is the product of the square of the root of the equation $6x^4 - 5x^2 - 30 = 0$?

(a) 25 (b) -5 (c) -25 (d) $5/6$

(15) What is the equation whose roots are 3i and 5i?

(a) $x^2 - 8x - 15 = 0$ (b) $x^2 + 8ix - 15 = 0$

(c) $x^2 - 15x + 8 = 0$ (d) None

(16) What is the equation whose roots are three times the roots of the equation $(x - 2)(x - 5) = 0$?

(a) $x^2 - 21x + 30 = 0$ (b) $x^2 - 7x + 10 = 0$

(c) $x^2 - 21x + 90 = 0$ (d) $x^2 - 81 = 0$

(17) The sum and product of the roots of the equation $x^2 + bx + c = 0$ are 18 and -115 respectively. What are the values of b and c?

(a) 18, 115 (b) 115, -18

(c) 18, -115 (d) -18, -115

(18) The sum of the roots of the equation $x^2 - 5x + q = 0$ is half the product of the roots of the equation $5x^2 + bx + m = 0$. What is the value of m?

(a) 25 (b) 50 (c) 12.5 (d) q

Chapter 8C

CUBIC EQUATIONS

CUBE ROOTS OF AN INTEGER:

(1) Cube Roots of 1:

Let x be the cube root of 1.

$$x^3 = 1$$

$$x^3 - 1 = 0$$

$$(x-1)(x^2 + x + 1) = 0$$

$$\text{either} \quad x - 1 = 0 \quad \text{or} \quad x^2 + x + 1 = 0$$

$$\Rightarrow \quad x = 1 \quad \text{or} \quad x = \frac{-1+\sqrt{3}\,i}{2}, \frac{-1-\sqrt{3}\,i}{2}$$

$$x = \omega, \omega^2$$

$$\text{where} \quad \omega = \frac{-1+\sqrt{3}\,i}{2} \quad \text{and} \quad \omega^2 = \frac{-1-\sqrt{3}\,i}{2}$$

All cube roots of 1 are 1, ω, ω^2.

Properties of ω:

(i) $\omega^3 = 1$

Proof:-

$$\omega^3 = \omega \,.\, \omega^2$$

$$= \left(\frac{-1+\sqrt{3}\,i}{2}\right)\left(\frac{-1-\sqrt{3}\,i}{2}\right)$$

$$= \frac{1+3}{4}$$

$$= 1$$

EXERCISE C-7

If $\omega = \dfrac{-1+\sqrt{3}\,i}{2}$ (ω is cube root of unity), then prove that

(1) $\omega^{25} + \omega^{12} = -\omega^2$

(2) $\omega^{26} + \omega^{48} + \omega^{37} = 0$

(3) $\omega^{38} + \omega^{13} = -1$

(4) $\omega^{6} + \omega^{8} = -\omega$

(5) $(1 + \omega^2)(\omega^5 + \omega^{14}) = -2$

(6) $(\omega^2 + \omega^{12})(\omega^{10} + \omega^{23}) = \omega$

(7) $\dfrac{\omega^7 + \omega^8}{1+\omega} = \omega$

(8) $\dfrac{\omega^{11} + \omega^{12}}{\omega^6 + \omega^{10}} = \omega^2$

Find all cube roots in terms of w (cube root of unity) of the following:

(9) 27

(10) -8

(11) 125

(12) -64

M.C.Q's C-7

If $\omega = \dfrac{-1+\sqrt{3}\,i}{2}$ (ω is cube root of 1), then

(1) $\omega^{12} + \omega^{15} = ?$

(a) ω^{27} (b) ω^2 (c) ω (d) 2

(2) $\omega^5 + \omega^{10} + \omega^{15} = ?$

(a) ω (b) $-\omega^2$ (c) 0 (d) 1

(3) $\omega^5 - \omega^{21} - \omega^{31} = ?$

(a) 2ω (b) $2\omega^2$ (c) 0 (d) 1

(4) $(-\omega^7 - \omega^8)\,.\,(\omega + \omega^2) = ?$

(a) -1 $\div$ (b) 1 (c) ω (d) ω^2

(5) $\{\omega + (\omega^{10} \div \omega^{14} + \omega^{14})^3\} = ?$

(a) 0 (b) $7 - \omega^2$ (c) $\omega + 1/2\,\omega^2$ (d) ω

(6) $\omega^5 \div \omega^7 \div \omega^2 \times \omega^{11} + \omega = ?$

(a) 2 (b) $2\omega^2$ (c) 2ω (d) 1

(7) $6\omega^4 - 2\omega^4 \times \omega^2 \div \omega - 2 = ?$

(a) 8ω (b) $\omega + 3$ (c) $2\omega^2 - 2$ (d) $2\omega^2$

(8) $(\omega + 1)^{10} \div (1 + \omega^2)^6 \times (-\omega - \omega^{2})^8 = ?$

(a) 1 (b) 0 (c) ω (d) ω^2

(9) $(1 + \omega)^2 \div (1 + \omega^2)^8 - (1 + \omega^{32})^5 = ?$

(a) $5\omega + 1$ (b) 3ω (c) $2\omega^2$ (d) $-\omega^2$

(10) $(\omega^{26} + \omega^{37})^8 \div (\omega^6)^7 = ?$

(a) 0 (b) ω (c) ω^2 (d) 1

ROOTS OF A CUBIC EQUATION

First root of the cubic equation $a_3x^3 + a_2x^2 + a_1x + a_o = 0$ can be found by factor theorem and then for other roots of the equation any method of the following can be used.

(1) Long Division.

(2) Synthetic Division Method

(3) Equating the Coefficients.

Example 2: Find all the roots of the cubic equation $2x^3 + x^2 - 6x - 4 = 0$.

Solution:-

Let, $f(x) = \overbrace{2x^3 + x^2 -}^{1} 8x - 4 = 0$

There is one variation of sign.

Therefore, there is only one positive real root. Now,

$$f(1) = -7 \neq 0$$
$$f(2) = 0$$

According to factor theorem 2 is a root. For other two roots we use above three method one by one.

(1) Long Division:

$x - 2$ is a factor.

$$\begin{array}{rl} & 2x^2 + 5x + 2 \quad \rightarrow \textit{quotient} \\ \textit{divisor} \leftarrow \quad x-2 & \overline{)\, 2x^3 + x^2 - 8x - 4} \quad \rightarrow \textit{dividend} \\ & \underline{\pm 2x^3 \mp 4x^2} \\ & \qquad 5x^2 - 8x - 4 \\ & \qquad \underline{\pm 5x^2 \mp 10x} \\ & \qquad\qquad 2x - 4 \\ & \qquad\qquad \underline{\pm 2x \mp 4} \\ & \qquad\qquad\quad 0 \quad \rightarrow \textit{remainder} \end{array}$$

$$\Rightarrow \quad (x - 2)(2x^2 + 5x + 2) = 0$$

either $\quad x - 2 = 0 \quad$ or $\quad 2x^2 + 5x + 2 = 0$

$$x = 2 \quad \text{or} \quad x = -2, \frac{-1}{2}$$

The three cube roots are -2, $-1/2$ and 2.

EXERCISE C-8

Find all roots of the following equations. Using any of the following method:

(i) Synthetic division method

(ii) Equating the coefficients

(iii) Long division method.

(1) $x^3 - 4x^2 - 25x + 28 = 0$ (2) $x^3 - 10x^2 + 19x + 30 = 0$

(3) $x^3 - 4x^2 + 5x - 20 = 0$ (4) $x^3 - 14x^2 + 28x + 120 = 0$

(5) $x^3 - 12x^2 + 41x - 42 = 0$ (6) $x^3 - 6x^2 + 9x - 54 = 0$

(7) One factor of the cubic expression $x^3 + 6x^2 - 85x - 450$ is $x + 5$. Find other two roots.

(8) One root of the cubic equation $x^3 - 32x^2 + 260x - 400$ is 2. Find other two roots.

(9) One quadratic factor of the expression $x^4 - 6x^3 + 17x^2 + 24x - 24$ is $x^2 + x - 2$. Find other quadratic factor.

(10) Find all the roots if two roots of the equation $x^4 + 4x^3 - 182x^2 - 186x + 360 = 0$ are 1 and -2.

(11) Using synthetic division method find the quotient and remainder when the polynomial $x^4 - 3x^3 + 2x^2 - 5x + 7$ is divided by $x - 2$.

(12) When $x^3 + px^2 - 3x - 30$ is divided by $x - 3$ the remainder is 42. Using synthetic division method, find the value of p.

(13) The equation $x^3 + 5x^2 - 17x - 21 = 0$ has a root -7. Find the other roots, using synthetic division method.

(14) The divisor and dividend are $x - 2$ and $2x^5 - 3x^3 + 2x^2 - 7$ respectively. Using synthetic division method, find the remainder and quotient.

(15) When the expression $3x^4 - px^2 - 2x + 9$ is divided by $x - 1$, the remainder is 5. Find the value of p, using synthetic division method.

(16) -2 and -3 are two factors of a polynomial equation $x^4 + 11x^3 + 41x^2 + 61x + 30 = 0$. Find other two roots, using synthetic division method.

(17) $x^2 - 1$ is a quadratic factor of the expression $x^4 + 9x^3 + 13x^2 - 9x - 14$. Find other quadratic factor, using synthetic division method.

M.C.Q's C-8

(1) What is the quotient if $x^3 - 2x^2 + 5x - 7$ is divided by $x - 1$?

(a) $x^2 + x - 1$ (b) $x^2 - x + 4$

(c) $x^2 - x + 1$ (d) $3x^2 + x + 1$

(2) What is the quotient if $x^3 - 5x^2 + 12$ is divided by $x - 2$?

(a) $x^2 - 3x$ (b) $x - 3$

(c) $x^2 - 3x - 6$ (d) $x^2 + 2x - 7$

(3) What is the quotient if $2x^3 - 8x + 5$ is divided by $x + 1$?

(a) $2x^2 - 10x$ (b) $2x^2 - 2x - 6$

(c) $2x^2 + 2x - 6$ (d) $2x^2 - 10$

(4) What is the quotient if the expression $3x^3 - 2x^2+7$ is divided by $x - 2$?

(a) $3x^2 + 4x + 8$ (b) $3x^2 + 4x + 15$

(c) $3x^2 + 4x$ (d) $3x^2 + 4x + 8$

(5) The expression $x^3 - 5$ is divided by $(x + 1)$. What is the quotient?

(a) $x^2 - 6x$ (b) $x^2 - 6$

(c) $x^2 + x + 1$ (d) $x^2 - x + 1$

Chapter 9

FACTOR AND REMAINDER THEOREMS

FACTOR THEOREM

$(x - a)$ is a factor of an expression $P(x)$ iff $p(a) = 0$.

Proof:

$P(x)$ can be written as

$$P(x) = (x - a)\,Q(x) + R$$

According to remainder theorem $R = P(a)$

$$P(x) = (x - a)\,Q(x) + P(a)$$

If $(x - a)$ is a factor of $P(x)$, the remainder must be zero.

$$R = 0$$
$$P(a) = 0$$

Example 1: Show that $(x - 2)$ is a factor of $P(x) = x^2 - 5x + 6$.

Solution:-

$$x = 2 \;=> P(2) = 2^2 - 5(2) + 6 = 0$$

By factor theorem $(x - 2)$ is a factor of $P(x)$.

Example 2: Prove that 3 is a root of the equation $x^2 - 5x + 6 = 0$.

Solution:-

Let $P(x) = x^2 - 5x + 6$

$$x = 3 \;=> P(3) = 3^2 - 5(3) + 6 = 0$$

Therefore, 3 is a root of the equation.

Example 3: Find the value of k if 5 is a root of the equation $x^2 - 7x + k = 0$.

Solution:-

Let $P(x) = x^2 - 7x + k = 0$

Since 5 is a root of the equation.

$$P(5) = 0$$
$$5^2 - 7(5) + k = 0$$
$$k = 10$$

REMAINDER THEOREM

$P(a)$ is the remainder if the expression $P(x)$ is divided by $x - a$.

Note: When a polynomial is divided by linear divisor $x - a$, the remainder can be found by remainder theorem. But we can not find quotient by this theorem.

Proof:

Case 1: $(x - a)$ is a linear divisor:

dividend = divisor × quotient + remainder

$$P(x) = (x - a) \,.\, Q(x) + R$$

Substituting $x = a$

$$P(a) = 0 \,.\, Q(x) + R = R$$

$$R = P(a)$$

Case 2: $(bx - a)$ is a linear divisor:

$$P(x) = (bx - a) \,.\, Q(x) + R$$

$$= b\left(x - \frac{a}{b}\right) . \, Q(x) + R$$

By substituting $x = \frac{a}{b}$, we get

$$R = P\left(\frac{a}{b}\right)$$

Note: Remainder and quotient can also be found by the following methods.

(1) Long Division

(2) Synthetic Division Method

Example 4: The expression $2x^3 - 3x^2 + x + 7$ is divided by $x - 3$. Find the remainder.

Solution:-

EXERCISE C-9

(1) $x^6 - 3x^2 + 9x + 2$ is divided by $x - 1$. Find the remainder.

(2) $5x^3 + 2x^2 - 6x - 7$ is divided by $x + 2$. Find the remainder.

(3) Find the remainder if $2x^4 - 6x^2 + 6x + 9$ is divided by $2x + 3$.

(4) Prove that $x^4 - 6x^3 + 17x^2 + 24x - 24$ is exactly divisible by $x - 3$.

(5) Is $(x + 2)$ a factor of $x^3 + 14x^2 - 3x + 1$.

(6) Is $(x - 6)$ a factor of $x^3 - 14x^2 + 28x + 120$.

(7) Is 8 a root of $x^3 - 16x^2 + 71x - 56 = 0$.

(8) Is 5 a root of $2x^4 - 3x + 9 = 0$.

(9) When the expression $2x^3 - bx^2 + 2x + 3$ is divided by $x - 2$, the remainder is 3. Find the value of b.

(10) $(2x - 1)$ is a factor of the polynomial $2x^3 + px^2 + 17x - 12$. Find the value of p.

(11) $2/5$ is a root of the equation $5x^3 - 7x^2 + mx + 12 = 0$. Find the value of m.

(12) The polynomial $x^4 - 6x^3 + px^2 + qx - 24$ is exactly divisible by $(x + 2)$ but leaves a remainder -24 when divided by $(x + 1)$. Find the value of p and q.

(13) The expression $x^3 + ax^2 + 41x - b$ leaves a remainder -180 when divided by $x + 2$ but exactly divisible by $(x - 2)$. Find the value of a and b.

M.C.Q's C-9

(1) $2x^5 - 3x^3 + 2x - 1$ is divided by $x + 1$. What is the remainder.

(a) 0 (b) 3 (c) -5 (d) -2

(2) When $2x^4 - 3x + 9k$ is divided by $x - 2$, the remainder is 8. What is the value of k?

(a) -2 (b) 5 (c) -6 (d) $-14/9$

(3) 4 is the root of the equation $x^3 - 4x^2 + px - 20 = 0$. What is the value of p?

(a) 2 (b) 5 (c) -37 (d) 6

(4) The expression $x^3 + px^2 - 25x + 28$ is divisible by $x - 1$. What is the value of p?

(a) 3 (b) 4 (c) -4 (d) -52

(5) The expression $x^4 + kx^3 + 9x^2 + 4x - 12$ is exactly divisible by $(x-2)^2$. What is the value of k?

(a) 6 (b) -6 (c) 7 (d) None

(6) The expression $x^4 + 4x^3 - mx^2 - 16x - 12$ is exactly divisible by $x^2 - 4$. What is the value of m?

(a) 3 (b) 1 (c) 2 (d) -5

(7) The expression $x^4 + kx^3 - 3x^2 + 11x - 6$ is exactly divisible by $x^2 - 2x + 1$. What is the value of k?

(a) 11 (b) 7 (c) -3 (d) None

(8) What number should be added to the expression $x^3 + 4x^2 + x - 8$, that the remainder must be 5, when the expression is divided by $x - 1$.

(a) -2 (b) 7 (c) -3 (d) 5

(9) When the express $2x^2 + 3x + 9$ is divided by $x - k$, the remainder is 9. What is the value of k?

(a) $-3/2$ (b) -5 (c) $1/5$ (d) $2/7$

(10) What number should be added to the equation $x^3 - 5x^2 + 7x + 2 = 0$ that 2 must be a root of the equation?

(a) 8 (b) 4 (c) -4 (d) none

(11) What number should be subtracted from the expression $x^3 + 2x^2 - 7x - 5$ that $x - 2$ must be a factor of the expression?

(a) -3 (b) 3 (c) 5 (d) 2

(12) What number should be added to the expression $2x^3 - 3x^2 + 2x + 10$ that $(x + 1)^2$ must be the factor of the expression?

(a) -11 (b) -1 (c) 5 (d) -3

Chapter 10

SYSTEM OF EQUATIONS OF TWO VARIABLES

The methods of solving the system of two variables polynomial equations of degree one or two are discussed in this chapter. Th ese equations can be solved by the following methods.

(1) Elimination method (2) Substitution Method

Example 1: Solve the system of equations using elimination method.

$$2x - 5y = 1 \quad \text{and} \quad 3x + 4y = -10$$

Solution:-

$$2x - 5y = 1 \longrightarrow (1)$$

$$3x + 4y = -10 \longrightarrow (2)$$

To eliminate y, multiply equation (1) by 4 and equation (2) by 5 and add.

$$\begin{aligned} 8x - 20y &= 4 \\ \underline{15x + 20y} &\underline{= -50} \\ 23x &= -46 \\ x &= -2 \end{aligned}$$

By substituting $x = -2$ in equation (1), we get

$$y = -1$$

Example 2: Solve the system of equations using substitution method.

$$y - x = 4 \quad \text{and} \quad 12x - 2y = 2$$

Solution:-

$$y - x = 4$$

$$y = x + 4 \longrightarrow (1)$$

$$12x - 2y = 2 \longrightarrow (2)$$

Substitute the value of y from equation (1) in equation (2)

$$12x - 2(x + 4) = 2 \Rightarrow x = 1$$

Substitute $x = 1$ in equation (1)

$$y = 1 + 4 \Rightarrow y = 5$$

Example 3: Solve the system of equations $x^2 - 3xy + y^2 = 0$ and $2x^2 - y^2 = 9$.

Solution:-

$$x^2 - 3xy + y^2 = 0 \longrightarrow (1)$$

$$2x^2 - y^2 = 9 \longrightarrow (2)$$

By factorizing equation (1), we get

$$(x - y)(x - 2y) = 0$$

either $x - y = 0$ or $x - 2y = 0$

$x = y \longrightarrow (3)$, $x = 2y \longrightarrow (4)$

Case 1: $x = y$ (3)

By substituting $x = y$ in equation (2), we get

$$y^2 = 9 \Rightarrow y = \pm 3$$

when $y = 3$, equation (3) gives $x = 3$

when $y = -3$, equation (3) gives $x = -3$

The value of (x, y) is (3, 3) or (−3, −3).

Case 2: $x = 2y$ (4)

By substituti ng in equation (2), we get $y = \pm \dfrac{3\sqrt{7}}{7}$.

when $y = \dfrac{3\sqrt{7}}{7}$, equation (4) gives $x = \dfrac{6\sqrt{7}}{7}$

when $y = \dfrac{-3\sqrt{7}}{7}$, equation (4) gives $x = \dfrac{-6\sqrt{7}}{7}$

The value of (x, y) is $\left(\dfrac{3\sqrt{7}}{7}, \dfrac{6\sqrt{7}}{7}\right)$ or $\left(\dfrac{-3\sqrt{7}}{7}, \dfrac{-6\sqrt{7}}{7}\right)$.

Since all four values of (x, y) satisfy equation (2). So the solution set is

$$\left\{(3, 3), (-3, -3), \left(\frac{3\sqrt{7}}{7}, \frac{6\sqrt{7}}{7}\right), \left(\frac{-3\sqrt{7}}{7}, \frac{-6\sqrt{7}}{7}\right)\right\}$$

Example 4: The age of Arif is twice the age of Kashif plus 5. The difference of their ages is 7. What are their ages.

Solution:-

Let Arif's age = x and Kashif's age = y

$$x = 2y + 5 \longrightarrow (1)$$

and $x - y = 7 \longrightarrow (2)$

By solving equation (1) and (2), we get

$$x = 9 \text{ and } y = 2$$

(24) The sum of the squares of two numbers is 617 and the difference of their squares is 105. Find the numbers.

(25) Find the two integers. The sum of twice first integer is increased by 2 and five more than twice second integer is 55. The difference of the squares of the numbers is 144.

(26) Ali sold 50 glasses of cold drinks. Lemonade sold for Rs.20 per glass and orangeade for Rs.25 per glass. The income is Rs.1165. How many glasses of each drink were sold?

(27) The sum of the surface areas of two spheres is 1348π cm^2 and the sum of their radii is 25 cm. Find the sum of the volumes of the spheres.

(28) Ali and Ahmed are two boys. The sum of their ages is 16. Three years ago Ali's age is there times the age of Ahmed. What are their ages now?

M.C.Q's C-10

(1) Mary buys some mango. Anne buys 5 less than twice the mango as Mary buys. How many mangoes Anne buys if sum of the mangoes is 7.

(a) 3 (b) 4 (c) 5 (d) 2

(2) Ali and Babar walk some metres. Ali walks three times Babar's walk decreased by 2. The quotient of Ali's and Babar's walk is 2.96. How many metres does Ali walk?

(a) 0.25 (b) 2.48 (c) 148 (d) 120

(3) The age of Talha is twice the age of Sarim plus 5. The difference of their ages is 7. What is the age of Sarim?

(a) 2 (b) 12 (c) 29 (d) 13

(4) Ali is x years old now. Four years ago his father's age is twice the age of Ali minus three and the sum of their ages is 49. What is the age of Ali's father now?

(a) 18 (b) 20 (c) 28 (d) 33

(5) Hina is x years old now. Five years times the age of her father is three times her age less two and the difference of their ages is 29. What is the age of her father now?

(a) 32 (b) 37 (c) 13 (d) 18

(6) Ali is five years older than Akbar. Akbar is 2 years younger than Kashif. What is the age of Akbar if the sum of their ages is 27.

(a) 20 (b) 12 (c) 10 (d) 15

(7) Six years times Ahsan's age will be three less than three times the age of Asghar. If Ahsan is 15 years old now. What is the age of Asgh ar?

(a) 3 (b) 7 (c) 12 (d) 2

(8) The sum of the ages of Ali and Talha is 48. Three years ago Ali is 50 minus two times the age of Talha. What is the age of Ali now?

(a) 24 (b) 32 (c) 36 (d) 40

(9) Mrs. Ali buys 45 fruits for Rs.160. He buys two types of fruits orange and banana. She boys orange and banana Rs.48 and Rs.36 per dozen respectively. What are the number of banana?

(a) 16 (b) 25 (c) 20 (d) 30

(10) In a cricket match 200 runs are made by 44 fours and sixes. What are the number of sixes?

(a) 16 (b) 29 (c) 12 (d) 14

[For more problems see "linear equation of one variable" and "ra te"].

COLLEGE MATHEMATICS WITH M.C.Q's

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