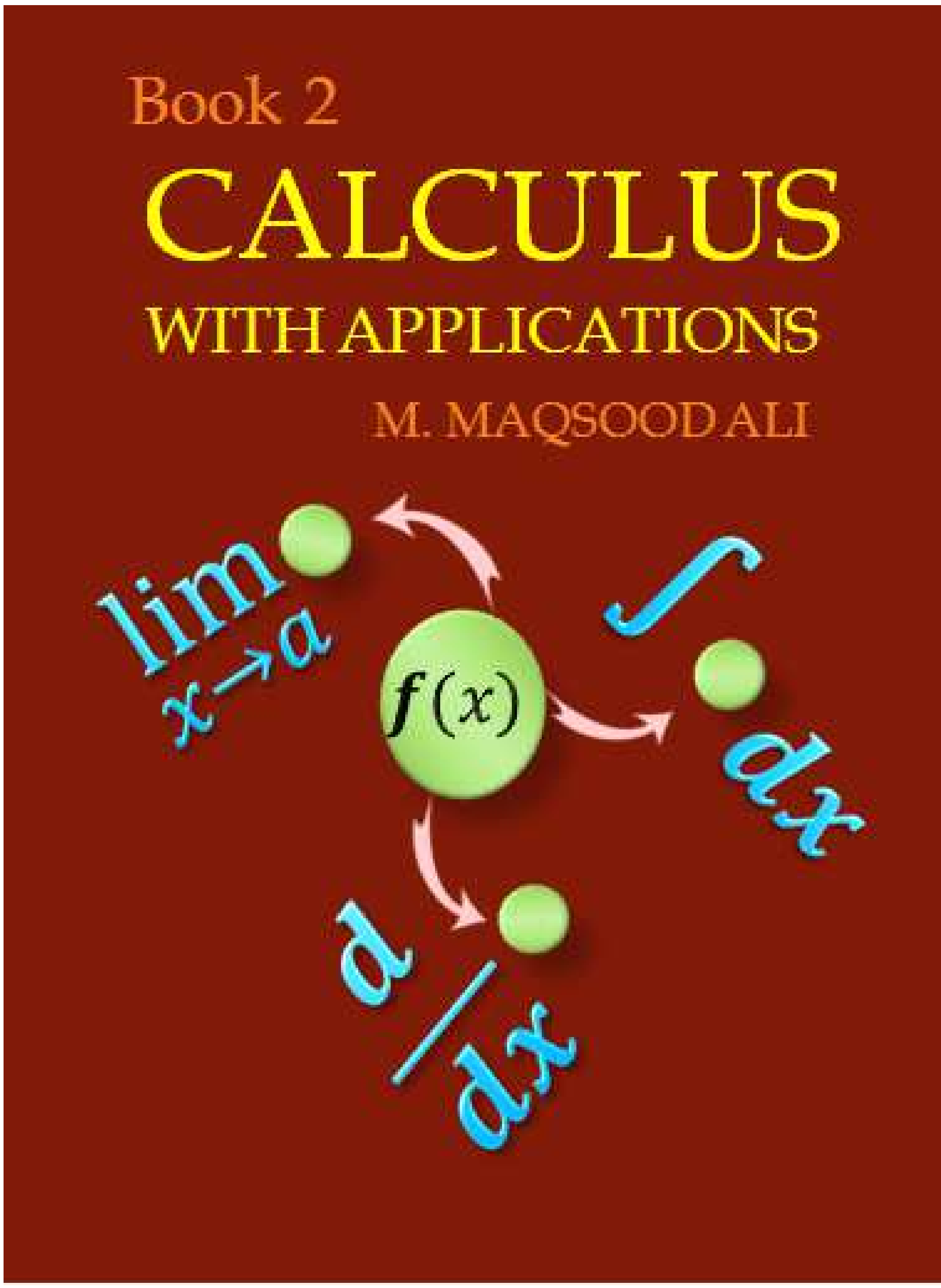
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
d/dx

IMPROPER INTEGRAL

The definite integral $\int_a^b f(x)\ dx$ is said to be improper integral if at least one limit is infinite. This type of improper integral is called improper integral of first kind.

There is another type of improper integral in which the limit of integration are finite but the integrand f becomes infinite at some points on the interval of integration. This type of improper integral is called improper integral of second kind.

IMPROPER INTEGRAL OF FIRST KIND:

CASE - I:

The improper integral of first kind in which upper limit of the integration is infinite is defined as

$$\int_a^{+\infty} f(x)\ dx = \lim_{t\to+\infty}\int_a^t f(x)\ dx$$

The improper integral is said to be convergent if the limit on the right side exist, otherwise it is said to be divergent.

Example 10.24:

$$\int_5^{+\infty}\frac{800\,dx}{x^2\sqrt{x^2-16}}=?$$

Solution:

$$\int_5^{+\infty}\frac{800\,dx}{x^2\sqrt{x^2-16}} = 800\lim_{t\to+\infty}\int_5^t\frac{dx}{x^2\sqrt{x^2-16}}$$

$$=\frac{400}{3}\lim_{t\to+\infty}\left[\frac{\sqrt{x^2-16}}{x}\right]_5^t$$

$$=\frac{400}{3}\lim_{t\to+\infty}\left\{\frac{\sqrt{t^2-10}}{t}-\frac{3}{5}\right\}$$

$$=\frac{400}{3}\lim_{t\to+\infty}\left\{\frac{t}{t}.\sqrt{1-16/t^2}-\frac{3}{5}\right\}$$

WHY $x \to +\infty$, WHY NOT $x = +\infty$

Explanation: If $x = +\infty$, then

$$\int_5^{+\infty} \frac{800\,dx}{x^2\sqrt{x^2-16}} = \frac{800}{3}\left[\frac{\sqrt{x^2-16}}{x}\right]_5^{+\infty}$$

$$= \frac{400}{3}\left[\frac{x}{x}.\sqrt{1-16/x^2}\right]_5^{+\infty}$$

$$= \frac{400}{3}\left\{\frac{\infty}{\infty} - \frac{3}{5}\right\}$$

$\frac{\infty}{\infty}$ is indeterminate form, so $x \to +\infty$.

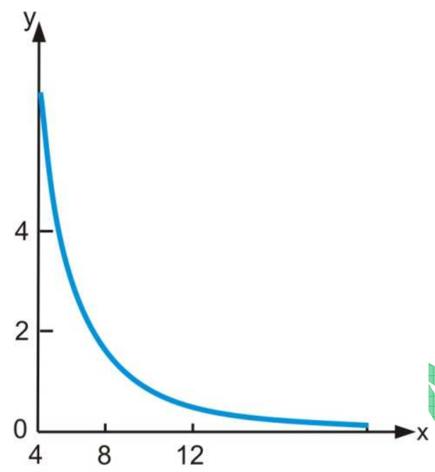

Figure *10.85*

Graph of the function

$f(x) = \dfrac{800}{x^2\sqrt{x^2-16}}$

from 5 to $+\infty$.

$$= \frac{400}{3}\lim_{t\to+\infty}\left\{\sqrt{1-16/t^2} - \frac{3}{5}\right\}$$

$$= \frac{400}{3}\left\{1 - \frac{3}{5}\right\}$$

$$= \frac{160}{3}$$

CASE - II:

The improper integral of first kind in which lower limit is infinite is defined as

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{t\to-\infty}\int_{t}^{b} f(x)\ dx$$

The improper integral is said to be convergent if the limit on the right side exist, otherwise it is said to be divergent.

Example 10.25:

$$\int_{-\infty}^{-5} \frac{500\,dx}{\sqrt{(x^2-9)^3}} = ?$$

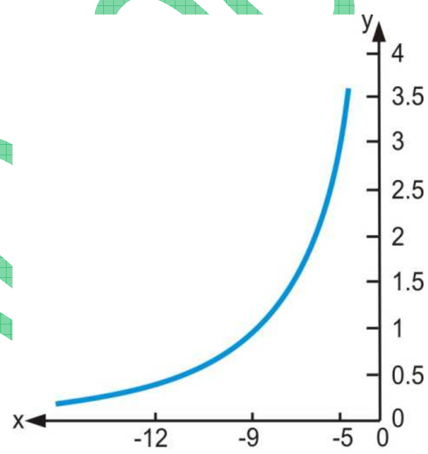

Figure 10.86

Graph of the function

$$f(x) = \frac{500\,dx}{\sqrt{(x^2-9)^3}}$$

from $-\infty$ to -5.

CASE - III:

The improper integral of first kind in which both the limits of integration are not finite is defined as

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{b} f(x)\ dx + \int_{-\infty}^{\infty} f(x)\ dx$$

$$= \lim_{p \to -\infty} \int_{p}^{b} f(x)\ dx + \lim_{q \to \infty} \int_{a}^{q} f(x)\ dx$$

The improper integral is said to be convergent if both limit of right side exist, otherwise it is said to be divergent.

Example 10.26:

$$\int_{-\infty}^{+\infty} \frac{900}{(x^2+25)^2}\, dx = ?$$

Solution:

$$\int_{-\infty}^{+\infty} \frac{900}{(x^2+25)^2}\, dx$$

$$= 900 \left\{ \int_{-\infty}^{0} \frac{dx}{(x^2+25)^2} + \int_{0}^{+\infty} \frac{dx}{(x^2+25)^2} \right\}$$

$$= 900 \left\{ \lim_{p\to-\infty} \int_{p}^{0} \frac{dx}{(x^2+25)^2} + \lim_{q\to+\infty} \int_{0}^{q} \frac{dx}{(x^2+25)^2} \right\} \to (1)$$

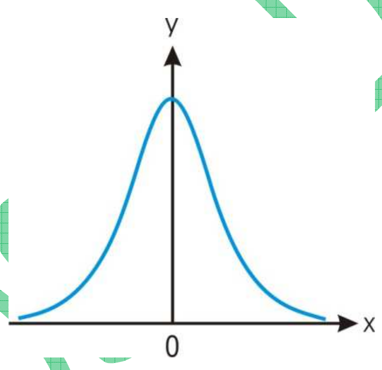

Figure 10.87

The graph of the curve $f(x) = \dfrac{900}{(x^2+25)^2}$

$= \lim_{p\to-\infty} \int_{p}^{0} \frac{dx}{(x^2+25)^2}$	$= \lim_{q\to+\infty} \int_{0}^{q} \frac{dx}{(x^2+25)^2}$
$= \frac{1}{250} \lim_{p\to-\infty} \left[\tan^{-1}\frac{x}{5} + \frac{5x}{x^2+25}\right]_p^0$	$= \frac{1}{250} \lim_{q\to+\infty} \left[\tan^{-1}\frac{x}{5} + \frac{5x}{x^2+25}\right]_0^q$

WHY : $x \to -\infty$, and $x \to +\infty$

WHY NOT $x = -\infty$ and $x = +\infty$

Explanation: If $x = -\infty$

$$\int_{-\infty}^{+\infty} \frac{900}{(x^2+25)^2}\,dx = \frac{900}{250}\left[tan^{-1}\frac{x}{5} + \frac{5x}{x^2+25}\right]_{-\infty}^{+\infty}$$

$$= \frac{18}{5}\left\{\frac{\pi}{2} + \frac{\infty}{\infty} + \frac{\pi}{2} - \frac{\infty}{\infty}\right\}$$

$\frac{\infty}{\infty}$ is indeterminate form.

So that

$$= -\frac{18}{5}\lim_{p\to-\infty}\left\{tan^{-1}\frac{p}{5} + \frac{5p}{p^2+25}\right\}$$

$$= -\frac{18}{5}\lim_{p\to-\infty}\left\{tan^{-1}\frac{p}{5} + \frac{p}{p^2}\left(\frac{p}{1+25/p^2}\right)\right\}$$

$$= -\frac{18}{5}\left\{-\frac{\pi}{2} + 0\right\}$$

$$= \frac{9\pi}{5}$$

$$= -\frac{18}{5}\lim_{p\to+\infty}\left\{tan^{-1}\frac{q}{5} + \frac{5q}{q^2+25}\right\}$$

$$= -\frac{18}{5}\lim_{q\to+\infty}\left\{tan^{-1}\frac{q}{5} + \frac{q}{q^2}\left(\frac{q}{1+25/q^2}\right)\right\}$$

$$= -\frac{18}{5}\left\{-\frac{\pi}{2} + 0\right\}$$

$$= \frac{9\pi}{5}$$

$$\int_{-\infty}^{+\infty} \frac{900\,dx}{(x^2+25)^2} = \frac{9\pi}{5} + \frac{9\pi}{5} = \frac{18\pi}{5}$$

IMPROPER INTEGRAL OF SECOND KIND:

CASE I:

The improper integral of second kind in which the integrand becomes infinite at any point on the interval of integration i.e $c \in [a, b]$, is defined as

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

$$\int_a^b f(x)\ dx = \lim_{p \to c^-} \int_a^p f(x)\ dx + \lim_{q \to c^+} \int_q^b f(x)\ dx$$

The improper integral from a to b convergent if the integral from a to c and c to b both converges otherwise integral from a to be divergent.

Example 10.27:

WHY NOT :

$$\int_0^4 \frac{dx}{(x-2)^{1/3}} = \frac{3}{2}\left[(x-2)^{2/3}\right]_0^4$$
$$= \frac{3}{2}\left\{2^{2/3} - 2^{2/3}\right\}$$
$$= 0$$

Explanation: $f(x)$ is not continuous at 2, but $f(x)$ is continuous on $[0,2) \cup (2,4]$

Figure 10.88

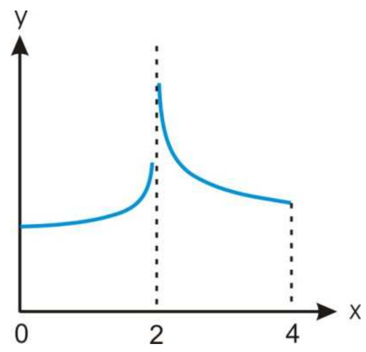

Figure 10.88

Area under the curve

$$f(x) = \frac{1}{(x-2)^{1/3}}$$

above x-axis between 0 and 4 is not equal to zero.

So that

$= \frac{3}{2}\lim_{p\to 2^-}\left[(x-2)^{2/3}\right]_0^p$	$= \frac{3}{2}\lim_{q\to 2^+}\left[(x-2)^{2/3}\right]_q^4$
$= \frac{3}{2}\lim_{p\to 2^-}\left\{(p-2)^{2/3} - 2^{\frac{2}{3}}\right\}$	$= \frac{3}{2}\lim_{q\to 2^+}\left\{2^{\frac{2}{3}} - (q-2)^{\frac{2}{3}}\right\}$
$= \frac{3}{2}(0 - 2^{2/3})$	$= \frac{3}{2}(2^{2/3} - 0)$
$= -\frac{3}{2}.2^{2/3}$	$= \frac{3}{2}.2^{2/3}$

So that

$$\int_0^4 \frac{dx}{(x-2)^{1/3}} = \frac{3}{2}.2^{2/3} + \frac{3}{2}.2^{2/3} = 3.2^{2/3}$$

Example 10.28:

$$\int_0^\pi sec^2x\ dx$$

Solution:

$f(x) = sec^2x \quad \Rightarrow \quad f\left(\frac{\pi}{2}\right) = \infty \quad ; \quad \frac{\pi}{2} \in (0,\pi)$

The value of integrand is infinite at $x = \pi/2 \in (0,\pi)$.

$f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

So that

$$\int_0^{\pi} sec^2x\ dx = \int_0^{\pi/2} sec^2x\ dx + \int_{\pi/2}^{\pi} sec^2x\ dx$$

$$= \lim_{P\to\frac{\pi}{2}^-} \int_0^{p} sec^2x\ dx + \lim_{q\to\frac{\pi}{2}^+} \int_q^{\pi} sec^2x\ dx$$

$$= \lim_{P\to\frac{\pi}{2}^-} [\tan x]_0^{\pi} + \lim_{q\to\frac{\pi}{2}^+} [\tan x]_q^{\pi} \quad \to (1)$$

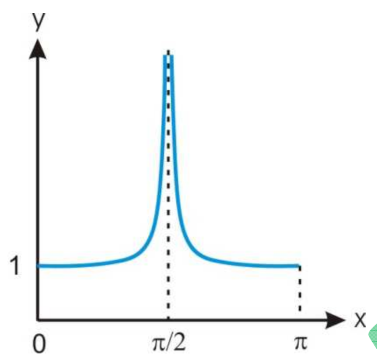

Figure 10.89

The graph of the function
$f(x) = sec^2x$
from 0 to π
shows that the area under
the curve is not equal to zero.

WHY NOT:

$$\int_0^{\pi} sec^2x\ dx = [\tan x]_0^{\pi} = \tan\pi - \tan 0 = 0$$

Its mean the area under the curve sec^2x between the vertical lines $x = 0$ and $x = \pi$ is zero but according to the **figure 10.89** the area under the curve is not zero.

Explanation:

$f(x) = sec^2x$ is not continuous at $\pi/2$.

So that

$\lim_{P\to\frac{\pi}{2}^-} [\tan x]_0^{p}$ $= \lim_{P\to\frac{\pi}{2}^-} \{\tan p - \tan 0\}$ $= \tan\pi/2$ $= \infty$	$\lim_{q\to\frac{\pi}{2}^+} [\tan x]_q^{\pi}$ $= \lim_{q\to\frac{\pi}{2}^+} \{\tan\pi - \tan q\}$ $= 0 - \tan\pi/2$ $= -\infty$

(1) becomes

$$\int_0^{\pi} sec^2x\ dx = \infty + \infty = \infty$$

CASE II:

The improper integral of second kind in which the integrand becomes infinite at lower limit “a” of the interval of integration [a, b] and continuous on [a, b] is defined as

$$\int_a^b f(x)\ dx + \lim_{c \to a^+} \int_c^b f(x)\ dx$$

The improper integral is said to be convergent if the limit on right side exist, otherwise it is said to be divergent.

Example 10.29:

$$\int_1^2 \frac{1}{(x-1)^{1/3}} dx$$

Solution:

The integrand is

$$f(x) = \frac{1}{(x-1)^{1/3}}$$

$$f(1) = \infty$$

The integrand $f(x)$ is discontinuous at $x = 1$ but continuous on (1,2].

$$\int_1^2 \frac{1}{(x-1)^{1/3}} dx = \lim_{c \to 1^+} \int_c^2 (x-1)^{-1/3}\ dx$$

WHY NOT: $x = 1$ or $x \to 1$, WHY: $x \to 1^+$
Explanation:
$f(x)$ is discontinuous at $x = 1, so\ x \neq 1\ but\ x \to 1$
Since $x \to 1\ means\ x \to 1^-\ or\ x \to 1^+\ \ and$
$1 \in [1^-, 2]\ but\ 1 \notin [1^+, 2], so\ x \to 1^+$
Figure 10.90

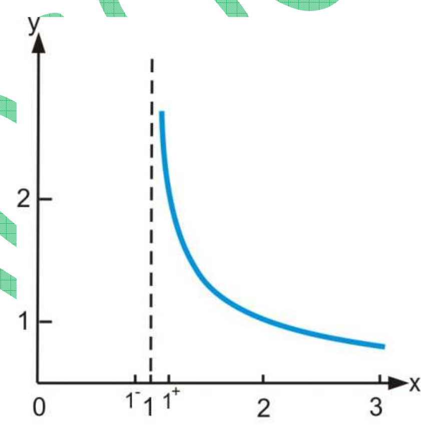

Figure 10.90

$$= \frac{3}{2} \lim_{c \to 1^+} \left[(x-1)^{2/3}\right]_c^2$$

$$= \frac{3}{2} \lim_{c \to 1^+} \left\{1 - (c-1)^{2/3}\right\}$$

$$= \frac{3}{2}\left\{1 - (1 - 1^{2/3}\right\}$$

$$= \frac{3}{2}$$

AUTHOR

M. MAQSOOD ALI
ASSISTANT PROFESSOR OF MATHEMATICS

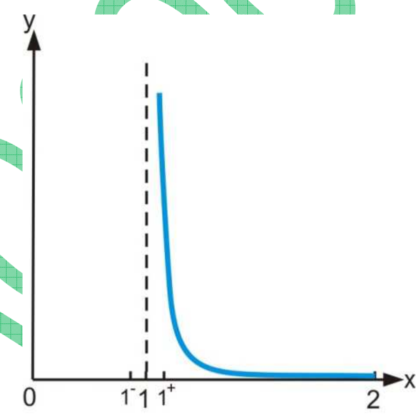

Figure 10.91

CASE III:

The improper integral of second kind in which the integrand becomes infinite at upper limit b on the interval of integration [a, b] and continuous on [a, b] is defined as

$$\int_{a}^{b} f(x)\ dx + \lim_{c\to b^-}\int_{a}^{c} f(x)\ dx$$

The improper integral is said to be convergent if the limit on right side exist, otherwise it is divergent.

Example 10.31:

$$\int_0^2 \frac{1}{2-x}dx$$

Solution:

The integrand is

$$f(x) = \frac{1}{2-x}$$
$$f(2) = \infty$$

The integrand $f(x)$ is discontinuous at $x = 2$ but continuous on $[0,2)$.

$$\int_0^2 \frac{1}{2-x}dx = \lim_{c\to 2^-}\int_0^c \frac{1}{2-x}\ dx$$

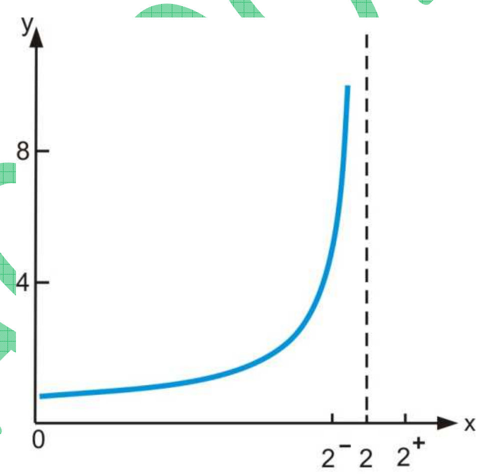

Figure 10.92

WHY NOT: $x = 2$ or $x \to 2$, WHY: $x \to 2^-$

Explanation:

$f(x)$ is discontinuous at $x = 2, so\ x \neq 2\ but\ x \to 2$

Since $x \to 2\ means\ x \to 2^-\ or\ x \to 2^+\ \ and$

$2 \in [0, 2^+]\ but\ 2 \notin [0, 2^-], so\ x \to 2^-$

Figure 10.92

$$= -\lim_{c\to 2^-}[\ln(2-x)]_0^c$$
$$= -\lim_{c\to 2^-}\{\ln(2-c) - ln2\}$$
$$= -\{\infty - ln2\}$$
$$= -\infty$$

The limit does not exist. So that improper integral

$$\int_1^2 \frac{1}{(x-1)^2}dx$$

diverges.

EXERCISE

Determine whether the following improper integrals converge or diverge if it converges find its value.

(1) $\int_{1}^{\infty} x^2 e^{-x^3}\, dx$

(2) $\int_{1}^{\infty} \frac{x}{x^2+1}\, dx$

(3) $\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2}-1}\, dx$

(4) $\int_{1}^{\infty} \frac{dx}{x(inx)^2}$

(5) $\int_{-\infty}^{1} \frac{1}{x^5}\, dx$

(6) $\int_{-\infty}^{2} \frac{2xdx}{(x^2-1)^2}$

(7) $\int_{-\infty}^{1} \frac{2x}{x^2-1}\, dx$

(8) $\int_{-\infty}^{\infty} x\, e^{-x^2}\, dx$

(9) $\int_{-\infty}^{\infty} \frac{xdx}{(x^2+9)^{3/2}}$

(10) $\int_{-1}^{1} \frac{1}{x}\, dx$

(11) $\int_{0}^{3} \frac{dx}{1-x^2}$

(12) $\int_{0}^{\pi} sec^2\, x\, dx$

(13) $\int_{0}^{\pi/2} sec^2\, x\, dx$