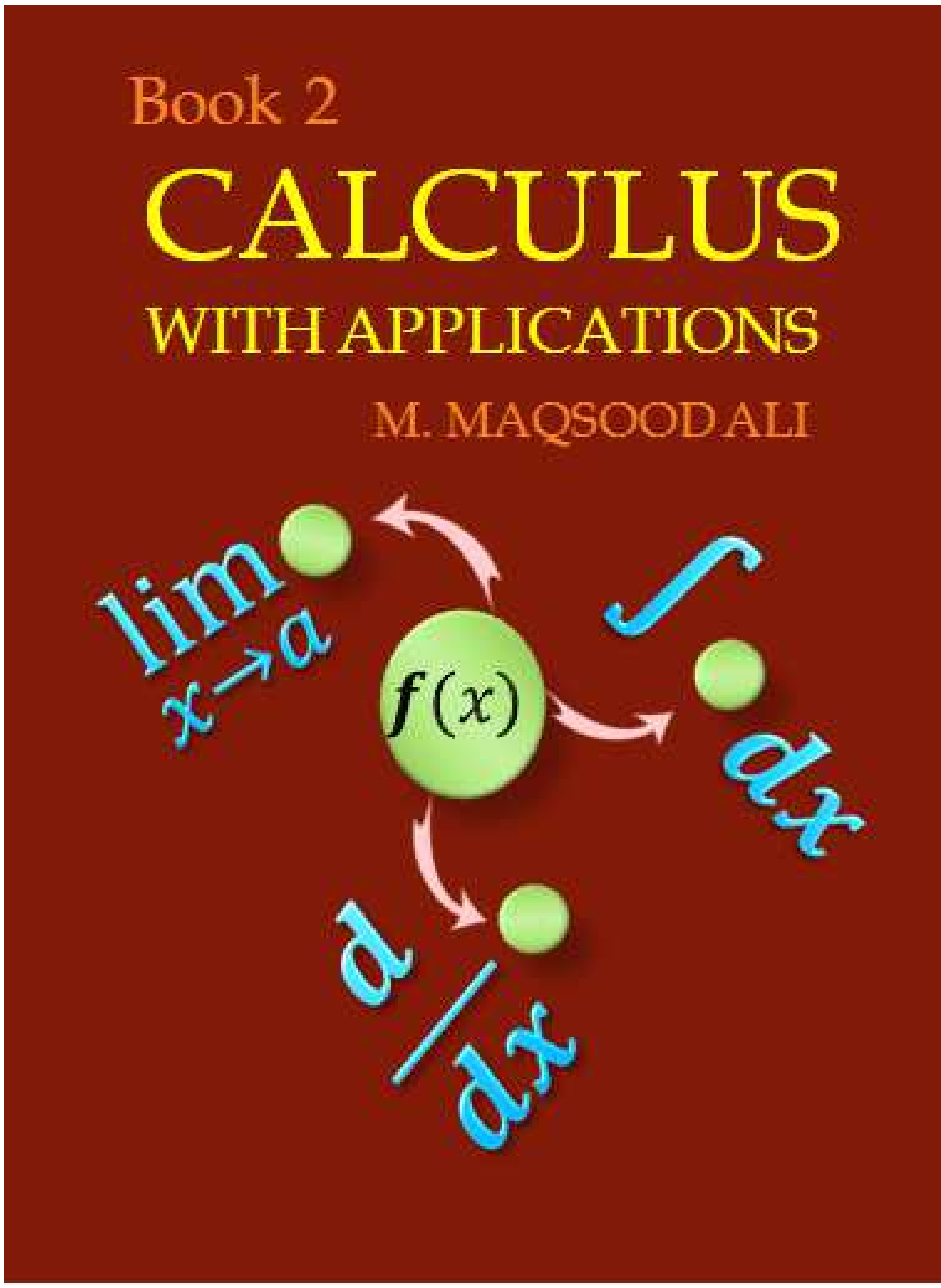
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

WALLI'S FORMULAE

$$\text{(i)} \int_0^{\frac{\pi}{2}} sin^n x \, dx = \frac{1.3.5 \ldots .(n-1)}{2.4.6 \ldots .n} \cdot \frac{\pi}{2} \quad ; \quad n \in E$$

$$= \frac{2.4.6 \ldots .(n-1)}{1.3.5 \ldots .n} \quad ; \quad n \in O$$

Proof:

$$a_n = \int_0^{\frac{\pi}{2}} sin^n x \, dx$$

We have

$$G_n = \int sin^n x \, dx = -\frac{1}{n} sin^{n-1} x \cos x + \frac{n-1}{n} G_{n-2}$$

$$\therefore \; a_n = \left| \frac{1}{n} sin^{n-1} x \cos x \right|_0^{\frac{\pi}{2}}$$

$$= 0 + \frac{n-1}{n} \left| G_{n-2} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{n-1}{n}\left[0 + \frac{n-3}{n-2} \cdot \left| G_{n-4} \right|_0^{\frac{\pi}{2}}\right]$$

$$= \frac{n-1}{n} \left| \frac{1}{n-2} sin^{n-3} x \cos x \right|_0^{\frac{\pi}{2}} + \frac{n-3}{n-2} \left| G_{n-4} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \left| G_{n-4} \right|_0^{\frac{\pi}{2}}$$

If n is even positive integer, then

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \, a_0$$

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} dx$$

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$a_n = \frac{1.3.5 \ldots \ldots (n-1)}{2.4.6 \ldots \ldots (n)} \cdot \frac{\pi}{2}$$

If n is odd positive integer, then

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{3} \cdot \frac{2}{1} a_1$$

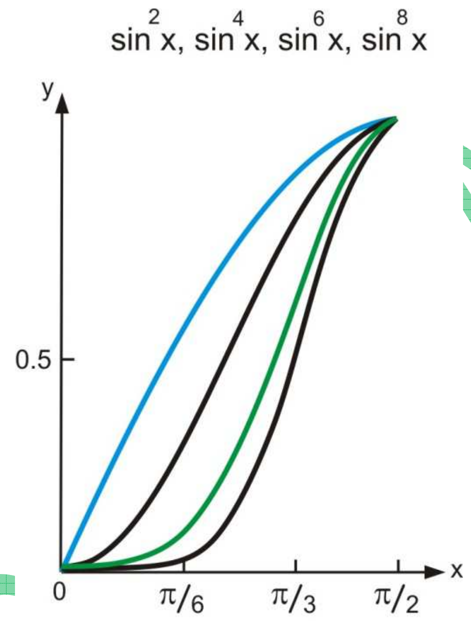

Figure *10.75*

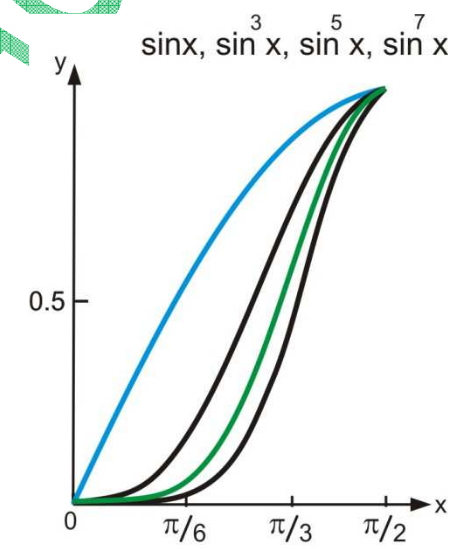

Figure *10.76*

$$a_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{3}\cdot\frac{2}{1}\int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{3}\cdot\frac{2}{1}\cdot 1$$

or

$$a_n = \frac{2.4.6\ldots\ldots(n-1)}{1.3.5\ldots\ldots n}$$

$$(ii)\quad \int_0^{\frac{\pi}{2}} cos^n x\,dx = \frac{1.3.5\ldots\ldots(n-1)}{2.4.6\ldots\ldots(n)}\cdot\frac{\pi}{2}\quad ;\quad n\in E$$

$$= \frac{2.4.6\ldots\ldots(n-1)}{1.3.5\ldots\ldots n}\quad ;\quad n\in O$$

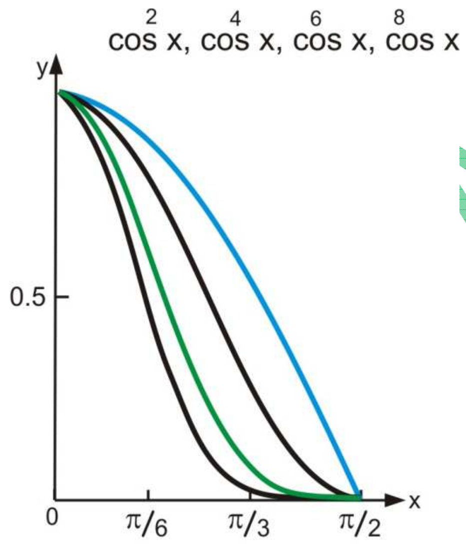

Figure ***10.77***

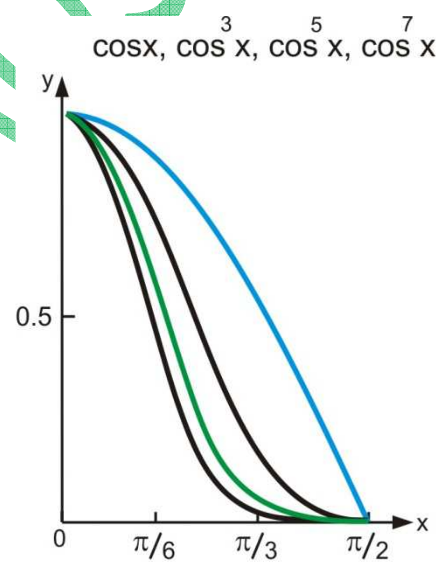

Figure ***10.78***

Proof

We have

$$G_n = \int cos^n x\,dx = \frac{1}{n}\,cos^{n-1} x\sin x + \frac{n-1}{n}\,G_{n-2}$$

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \frac{1}{n}\left|\,\cos^{n-1} x sinx\right|_0^{\frac{\pi}{2}} + \frac{n-1}{n}\left|G_{n-2}\right|_0^{\frac{\pi}{2}}$$

$$= 0 + \frac{n-1}{n}\left[\frac{1}{n-2}\left|cos^{n-3} x\ sinx\right|_0^{\frac{\pi}{2}} + \frac{n-3}{n-2}\left|G_{n-4}\right|_0^{\frac{\pi}{2}}\right]$$

$$= \frac{n-1}{n}\left[0 + \frac{n-3}{n-2}\left|G_{n-4}\right|_0^{\frac{\pi}{2}}\right]$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\left|G_{n-4}\right|_0^{\frac{\pi}{2}}$$

If n is even positive integer, then

$$a_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\left|G_0\right|_0^{\frac{\pi}{2}}$$

$$a_n = \frac{1.3.5\ldots\ldots(n-1)}{2.4.6\ldots\ldots(n)}\cdot\frac{\pi}{2}$$

If n is odd positive integer, then

$$a_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{6}{5}\cdot\frac{4}{3}\cdot\frac{2}{1}\left|G_1\right|_0^{\frac{\pi}{2}}$$

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \dots \frac{6}{5} \cdot \frac{4}{3} \cdot \frac{2}{1}\, 1$$

$$a_n = \frac{2.4.6 \dots .n}{1.3.5 \dots .(n-1)}$$

So that

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{1.3.5 \dots (n-1)}{2.4.6 \dots (n)} \cdot \frac{\pi}{2}\ ; \ n \in E$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{2.4.6 \dots .n}{1.3.5 \dots .(n-1)}\ ; n \in O$$

Another Reduction Formula For Limit 0 to $\frac{\pi}{2}$:

$$(iii)\ a_{m,n} = \int_0^{\frac{\pi}{2}} sin^m \text{x}\, cos^n \text{x}\, dx$$

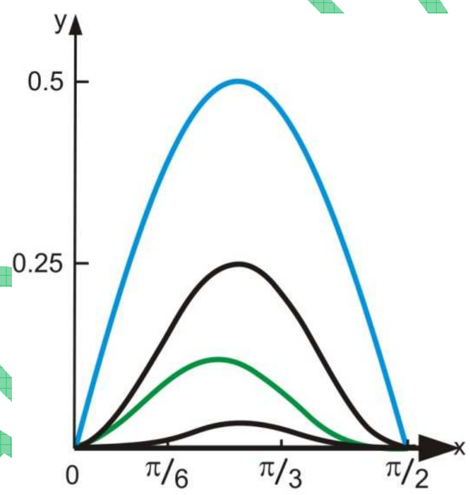

Figure ***10.79***

Curves $\boldsymbol{sinx\, cosx, sin^2x\, cos^2x, sin^3x\, cos^4x\, , sin^5x\, cos^4x}$

We know that
If $G_{m,n} = \int sin^m x\, cos^n x\, dx$ reduction formula which reduce n is given by

$$G_{m,n} = \frac{1}{m+n} sin^{m+1} x\, cos^{n-1} x + \frac{n-1}{m+n} G_{m,n-2}$$

So that

$$a_{m,n} = \frac{1}{m+n} \left| sin^{m+1} x\, cos^{n-1} x \right|_0^{\frac{\pi}{2}} + \frac{n-1}{m+n} G_{m,n-2}$$

$$a_{m,n} = \frac{n-1}{m+n} a_{m,n-2} \qquad \rightarrow (i)$$

We also have the reduction formula which reduce m

$$G_{m,n} = \frac{-1}{m+n} sin^{m-1} x\, cos^{n+1} x + \frac{m-1}{m+n} G_{m-2,n}$$

$$a_{m,n} = \frac{-1}{m+n} \left| sin^{m-1} x\, cos^{n+1} x \right|_0^{\frac{\pi}{2}} + \frac{m-1}{m+n} G_{m-2,n}$$

$$a_{m,n} = \frac{m-1}{m+n} G_{m-2,n}$$

If n and m both are even:

Using alternating eq. (i) and (ii)

$$a_{m,n} = \frac{n-1}{m+n} \cdot \frac{m-1}{m+n-2} a_{m-2,n-2}$$

$$a_{m,n} = \frac{n-1}{m+n} \cdot \frac{m-1}{m+n-2} \cdot \frac{n-3}{m+n-4} \cdot \frac{m-3}{m+n-6} \cdots a_{0,0}$$

Now

$$a_{0,0} = \int_0^{\frac{\pi}{2}} sin^0\, x\, cos^0\, x\, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

Put in (iii)

$$a_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots\ldots(m-1)(m-3)\ldots}{(m+n)(m+n-2)(m+n-4)\cdots} \frac{\pi}{2}$$

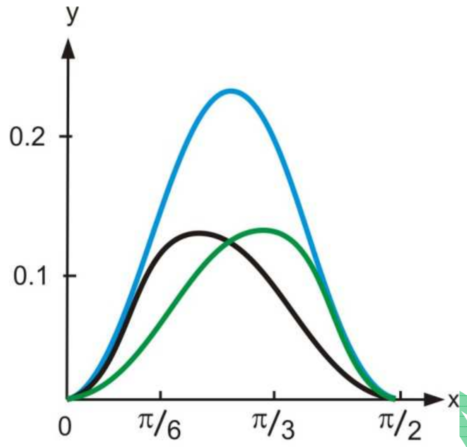

Figure 10.80

Curves $sin^2x\, cos^2x$, $sin^2x\, cos^4x$, $sin^4x\, cos^2x$

If n is even and m is odd:

$$a_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots\ldots(m-1)(m-3)(m-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots\ldots} a_{0,1}$$

Now

$$a_{0,1} = \int_0^{\frac{\pi}{2}} sin\, x\, cos^0\, x\, dx$$

$$a_{0,1} = \int_0^{\frac{\pi}{2}} sin\, x\, dx = |-\cos x\,|_0^{\frac{\pi}{2}} = -\left(cos\frac{\pi}{2} - cos0\right)$$

$$a_{0,1} = -(0-1)$$

$$a_{0,1} = 1$$

So

$$a_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots\ldots(m-1)(m-3)(m-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots\ldots}$$

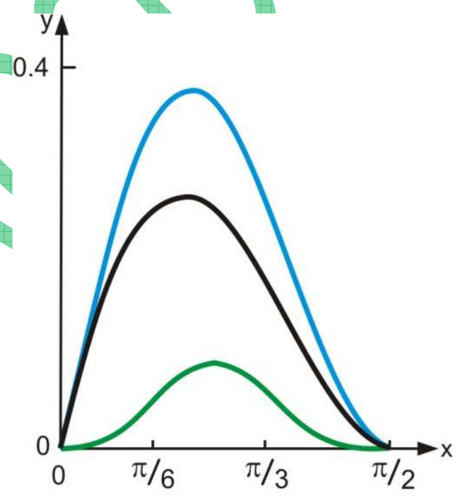

Figure 10.81

Curves $sinx\, cos^2x$, $sinx\, cos^4x$, $sin^3x\, cos^4x$

If n is odd and m is even:

In this case formula (v) is valid.

Example 10.23:

A design is painted on the door of a toy car, as shown in the **figure 10.82**, in two colors blue and gray. The upper part is blue and lower part gray. The curves which form the design are

$f(x) = sinx,\ g(x) = sin^5x$ and $h(x) = sin^7x$ between $x = 0$ and $x = \frac{\pi}{2}$, top to bottom respectively.

(a) Using Walli's formula find the areas of blue and gray colors.

Physical science:

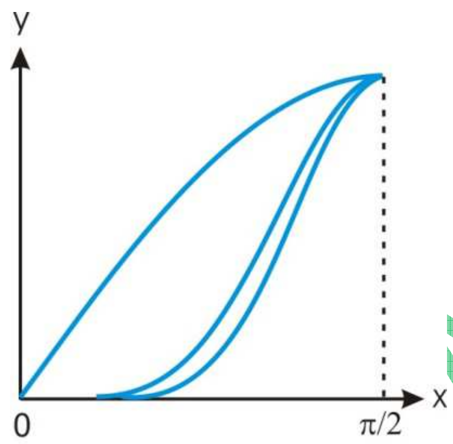

Figure 10.82

$$= \int_0^{\pi/2} sinx dx - \int_0^{\pi/2} sin^5 x dx$$

$$= [sinx]_0^{\pi/2} - \frac{2.4}{1.3.5}$$

$$= 1 - \frac{8}{15}$$

$$= \frac{7}{15} \text{ sq.units}$$

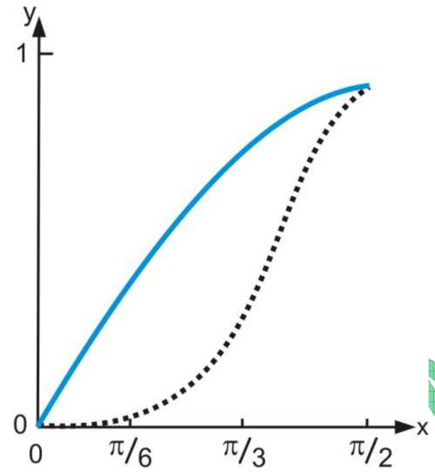

Figure ***10.83***

Area between the curves *g(x)* and *h(x)* .

$$A_2 = \int_0^{\pi/2} \{g(x) - h(x)\} dx$$

$$= \int_0^{\pi/2} g(x) dx - \int_0^{\pi/2} h(x) dx$$

$$= \int_0^{\pi/2} sin^5 x dx - \int_0^{\pi/2} sin^7 x dx$$

$$= \frac{2.4}{1.3.5} - \frac{2.4.6}{1.3.5.7} = \frac{8}{15} - \frac{16}{35}$$

$$= \frac{8}{105} \, sq.units$$

.

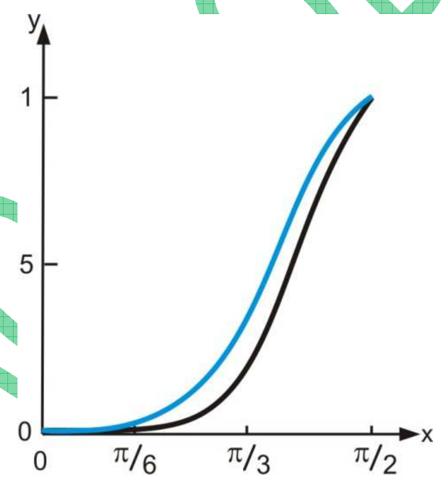

Figure ***10.84***

EXERCISE

Evaluate the following:

(1) $\int_0^{\frac{\pi}{2}} sin^3 x \cos x \, dx$ (2) $\int_0^1 x(x^2 - 1)^2 \, dx$

(3) $\int_0^{\frac{\pi}{2}} sin\, 2x \cos 3x \, dx$ (4) $\int_0^1 x\, e^x \, dx$

(5) $\int_0^{\frac{\pi}{2}} (e^x + e^{-x}) \, dx$ (6) $\int_0^{\frac{\pi}{2}} sin^{10} x \, dx$

(7) $\int_0^{\frac{\pi}{2}} sin^7 x\, cos^5\, x \, dx$ (8) $\int_0^{\frac{\pi}{2}} cos^7 x \, dx$

(9) $\int_0^{\frac{\pi}{2}} tan^5 x \, dx$ (10) $\int_0^{\frac{\pi}{2}} sin^8 x\, cos^4\, x \, dx$

(11) $\int_0^1 (in\, x)^7 \, dx$