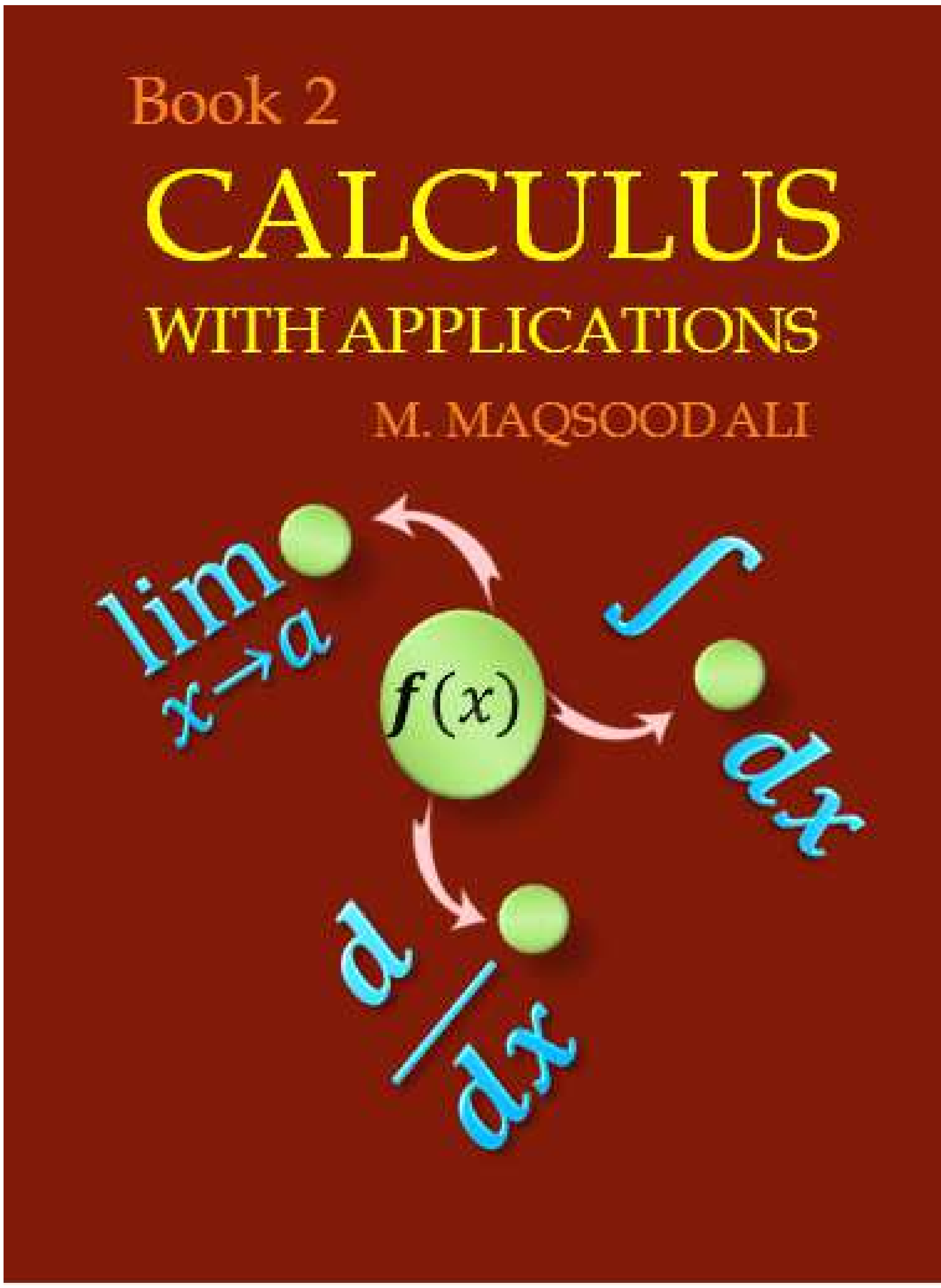
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

## TYPES OF DEFINITE INTEGRATIONS:

There are two types of definite integration

(i) Proper integration.

(ii) Improper integration.

## PROPER INTEGRATION:

The definite integral $\int_a^b f(x)$ is said to be proper integral if

(i) Both the limits $a$ and $b$ are finite.

(ii) The integrand $f$ is finite at every point on the interval of integration.

### FORMULAE

### AREA, VOLUME, SURFACE AREA AND ARC LENGTH:

Formulae for Area under the curve, Volume of solid obtained by rotating a curve, Surface area of the solid and Arc length are given below.

### (1) AREA UNDER THE CURVE:

Area under the curve $f(x)$ above x-axis between the vertical lines $x = a$ and $x = b$, as shown in the **figure 10.53** is

$$Area = \int_a^b f(x)dx$$

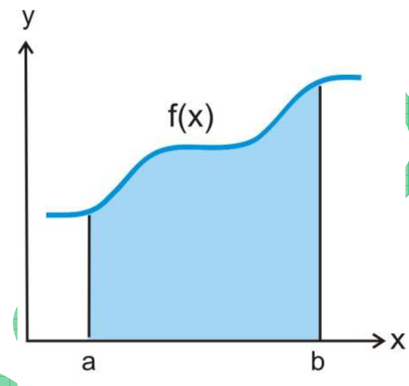

**Figure 10.53**

### (2) VOLUME OF A SOLID BY ROTATING A REGION:

The graph of $f(x)$ and x-axis form a region $R$ between vertical lines $x = a$ and $x = b$, as shown in the **figure 10.54**

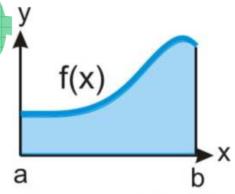

**Figure 10.54**

(a) Volume of solid generated by rotating region $R$ about x-axis, as shown in **figure 10.55** is

$$V = \pi \int_a^b [f(x)]^2 dx$$

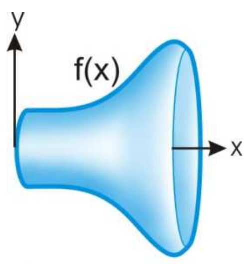

**Figure 10.55**

(b) Volume of solid generated by rotating region $R$ about y-axis, as shown in **figure 10.56** is

$$V = 2\pi \int_a^b x f(x) dx$$

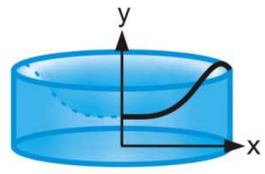

**Figure 10.56**

**(3) LENGTH OF A PLANE CURVE:**

If $f(x)$ is a curve continuous on $[a, b]$, the length of the curve from $x = a$ and $x = b$, as shown in the **figure 10.57** is

$$L = \int_a^b \sqrt{1 + [f(x)]^2} dx$$

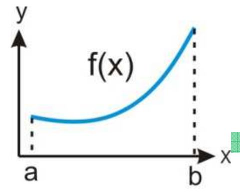

**Figure 10.57**

**(4) SURFACE AREA OF A SOLID:**

Surface area of solid generated by rotating the curve $f(x)$ about x-axis between vertical lines $x = a$ and $x = b$, as shown in **figure 10.58** is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f(x)]^2} dx$$

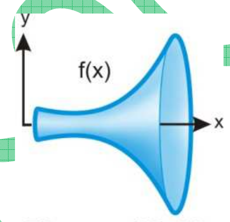

**Figure 10.58**



**Example 10.14:** **Physical Science**

A solid of revolution is created by revolving a region bounded by the graph

$$f(x) = x^3 - 6x + 20$$

and $x$-axis between the vertical lines $x = 0$ and $x = 5$ cm, **figures 10.59** and **10.60**.

(a) Find the volume of the solid.

(b) The density of the copper (metal) is 8940 $\text{kg/m}^3$, which is used to create the solid. How much metal is used?

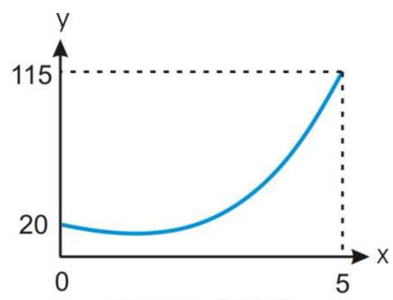

**Figure 10.59**

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Physical Science

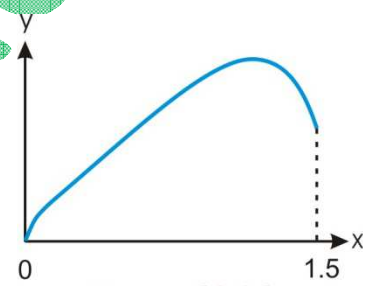


Figure 10.61

Using integration by parts

$$V = \frac{\pi}{4}\left[e^{2x}(sin2x - cos2x)\right]_0^{3/2}$$

$$= 5.93\,\pi\,units^3.$$

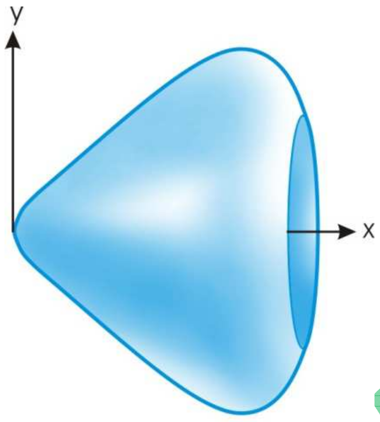

Figure 10.62

**Example 10.16:** Physical Science

Find the volume of the solid obtained by revolving the region bounded by the graph

$$f(x) = x^2 sinx^2 + 4$$

and $x$-axis between the vertical lines $x = 0$ and $x = \pi/2$ about y-axis, **figures 10.63** and **10.64**.

**Solution:**

$$f(x) = x^2 sinx^2 + 4$$

The volume of the solid is

$$V = 2\pi\int_0^{\pi/2} x\{x^2\,sinx^2 + 4\}\,dx$$

$$= 2\pi\,\{\int_0^{\pi/2} x.x^2\,sinx^2 + \int_0^{\pi/2} 4xdx\,\}$$

$$= 2\pi\int_0^{\pi/2}(x^2\,sinx^2)xdx + \pi^3$$

Let $u = x^2$

$$\frac{1}{2}\,du = xdx$$

$$x = 0 \Rightarrow u = 0$$

$$x = \frac{\pi}{2} \Rightarrow u = \frac{\pi^2}{4}$$

$$V = \pi\int_0^{\pi^2/4} u\,sinu\,du + \pi^3$$

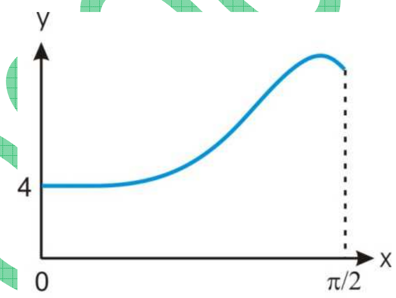

Figure 10.63

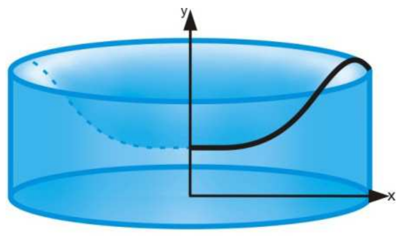

Figure 10.64

$= \pi\,[-u\,cosu + sinu]_0^{\pi^2/4} + \pi^3$

$= 39\,units^3$

**Example 10.17:**

**Physical Science**

Find the volume of hollow solid of revolution of thickness 2 cm created by revolving the region bounded by the graph $f(x) = x + 5$ and $g(x)$ between $x = 0$ cm and $x = 10$ cm about $x$ −axis, **figures 10.65** and **10.66.**

(a) Gold which is used to create the solid has density 19.32 gram/cm$^3$.How much mass of gold is used?

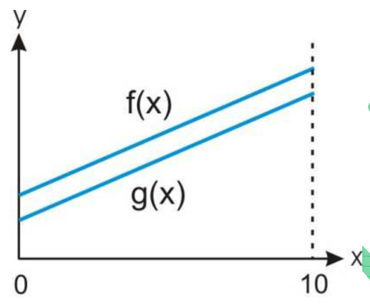

**Figure 10.65**

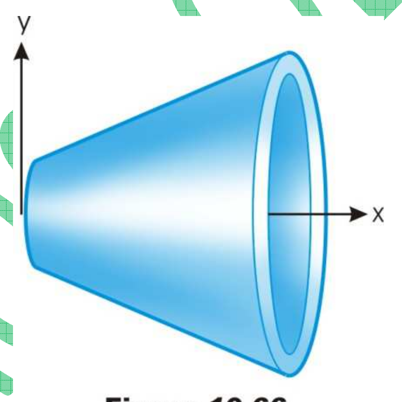

**Figure 10.66**

**Solution**

$$f(x) = x + 5$$

The thickness is 2 cm, so that

$$g(x) = f(x) - 2 = x + 5 - 2 = x + 3$$

The volume of hollow solid

$$V = \int_0^{10} \{[f(x)]^2 - [g(x)]^2\}dx$$

$$= \int_0^{10} (4x + 16)\,dx$$

$$= [2x^2 + 16x\,]_0^{10}$$

$$= 360 \text{ cm}^3$$

(a) Density of gold= 19.32 gm/cm$^3$

The mass of the gold used to create solid

$= 360 \times 19.32$

$= 6955.2$ grams

$= 6.96$ kg

**Example 10.18:** **Physical Science**

A solid of thickness 1cm is created by revolving the region bounded by the graphs of

$$f(x) = x^2 - 8x + 26 \text{ and } g(x)$$

between $x = 0$ cm and $x = 10$ cm about x-axis. Find the volume of the solid of base 2cm, **figures 10.67** and **10.68**.

**Solution:**

$$f(x) = x^2 - 8x + 26$$

The thickness of the solid is 1 cm, so that

$$g(x) = f(x) - 1 = x^2 - 8x + 25$$

The volume of solid is the sum of two solids one is created by revolving the region bounded by *f(x)* and *x*-axis between *x = 0* cm and *x = 2* cm about *x*-axis.

$$V_1 = \pi \int_0^2 [f(x)]^2\, dx = \pi \int_0^2 (x^2 - 8x + 26)^2\, dx$$

$$= \pi \int_0^2 (x^4 - 16x^3 + 116x^2 - 416x + 676)\, dx$$

$$= \pi[\frac{x^5}{5} - 4x^4 + \frac{116x^3}{3} - 208x^2 + 676x]_0^2$$

$$= 771.73\,\pi \text{ cm}^3.$$

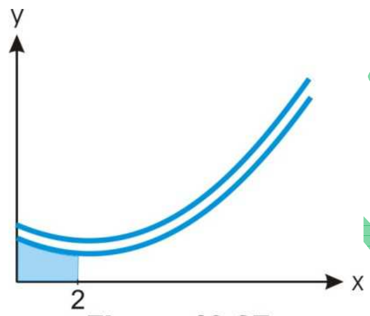

**Figure 10.67**

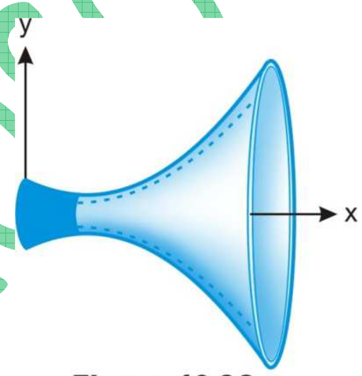

**Figure 10.68**

The other is hollow solid of thickness 1 cm created by the region bounded by the graphs *f(x)* and *g(x)* between *x = 2* cm and *x = 10* cm.

$$V_2 = \pi \int_2^{10} \{[f(x)]^2 - [g(x)]^2\}\, dx$$

$$= \pi \int_2^{10} (2x^2 - 16x + 51)\, dx$$

$$= \pi[\frac{2x^3}{3} - 8x^2 + 51x]_2^{10}$$

$$= 301.34\pi \text{ cm}^3.$$

Total volume of the solid

$$V = V_1 + V_2$$
$$= 771.73\pi + 301.34\pi$$
$$= 1073.07\pi \text{ cm}^3.$$

**Example 10.19:** **Physical Science**

A solid of revolution is obtained revolving a region bounded by the graph $f(x) = \sqrt{25 - x^2}$ and $x$-axis between $x = 0$ and $x = 4$ feet, figures 10.69 and 10.70.

(a) Find the curved surface area of the solid.

(b) Find the total surface area of the solid.

**Solution:**

$$f(x) = \sqrt{25 - x^2}$$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

The curved surface area of the solid

$$A_1 = 2\pi \int_a^b f(x).\sqrt{1 + [f(x)]^2}\ dx$$

$$= 2\pi \int_0^4 \sqrt{25 - x^2}\ .\ \sqrt{1 + \frac{x^2}{25 - x^2}}\ dx$$

$$= 2\pi \int_0^4 \sqrt{25 - x^2}\ .\ \frac{5}{\sqrt{25 - x^2}}\ dx$$

$$= 10\pi\ dx \int_0^4 dx$$

$$= 10\pi\ [x]_0^4$$

$$= 10\pi\ (4 - 0)$$

$$= 40\pi \text{ sq. feet.}$$

Total surface area is

$$A = \pi {r_1}^2 + \pi r_2^2 + 40\pi$$

$$= \pi(5)^2 + \pi(3)^2 + 40\pi$$

$$= 74\pi \text{ square feet.}$$

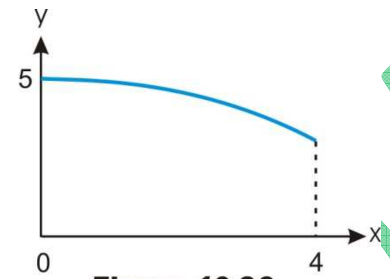

**Figure 10.69**

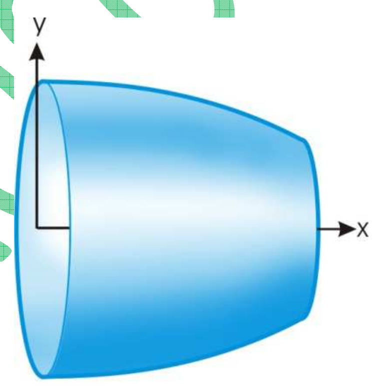

**Figure 10.70**

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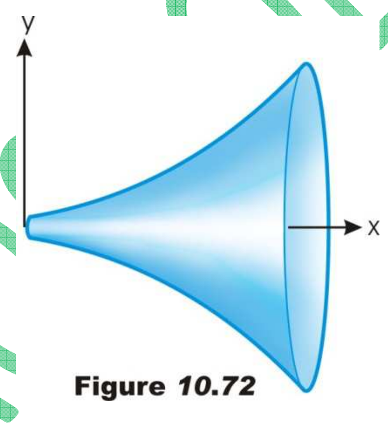


Figure 10.72

Physical Science

$$= \int_0^{20} \sqrt{1 + \frac{4}{9}x} \quad dx$$

$$= \frac{3}{2} \; [(1 + \frac{4}{9}x)^{3/2}]_0^{20}$$

$$= 45 \text{ units.}$$

**Example 10.22:**

**Physical Science**

The graph of $f(x) = 3x^3$ between vertical lines $x = 0$ and $x = 1$ cm is a curve, **figure 10.73**.

(a) Find the volume of solid revolving the region bounded by $f(x)$, $y = 0$ and vertical lines $x = 0$ and $x = 1$, **figure 10.74**.

(b) The density of the silver (metal) is 10.49 $\text{gm/cm}^3$ which will use to create the solid. How much metal will be used?

(c) The paint covers ________ per litre, which will use to paint the solid. How much paint will be used to paint whole solid?

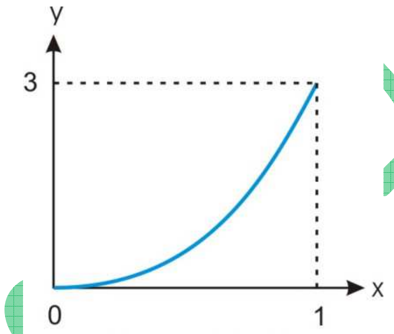

**Figure 10.73**

**Solution:**

(a) $f(x) = 3x^3$

The volume of the solid between $x = 0$ and $x = 1$ cm.

$$V = \pi \int_0^1 [3x^3]^2 \quad dx = 9\pi \int_0^1 x^6 \; dx$$

$$= \frac{9}{7}\pi . [x^7]_0^1$$

$$= \frac{9\pi}{7} \text{ cm}^3 = 4.04 \text{ cm}^3.$$

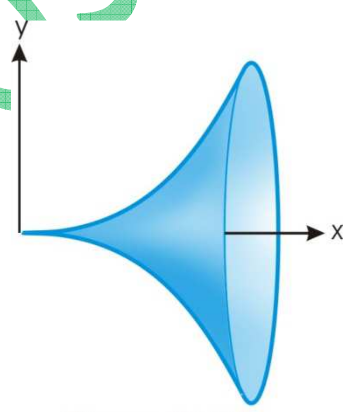

**Figure 10.74**

(b) Density of silver= 10.49 $\text{gm/cm}^3$

The silver used to create the solid

$$= 4.04 \times 10.49$$

$$= 42.38 \text{ grams}$$

(c) $f(x) = 3x^3$

$f'(x) = 9x^2$

The curved surface area of the solid between $x = 0$ and $x = 1$

$$S = 2\pi \int_0^1 3x^3 . \sqrt{1 + 81x^4} \, dx$$

$$= \frac{\pi}{81} [(1 + 81x^4)^{3/2}]_0^1$$

$$= 28.76.$$

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