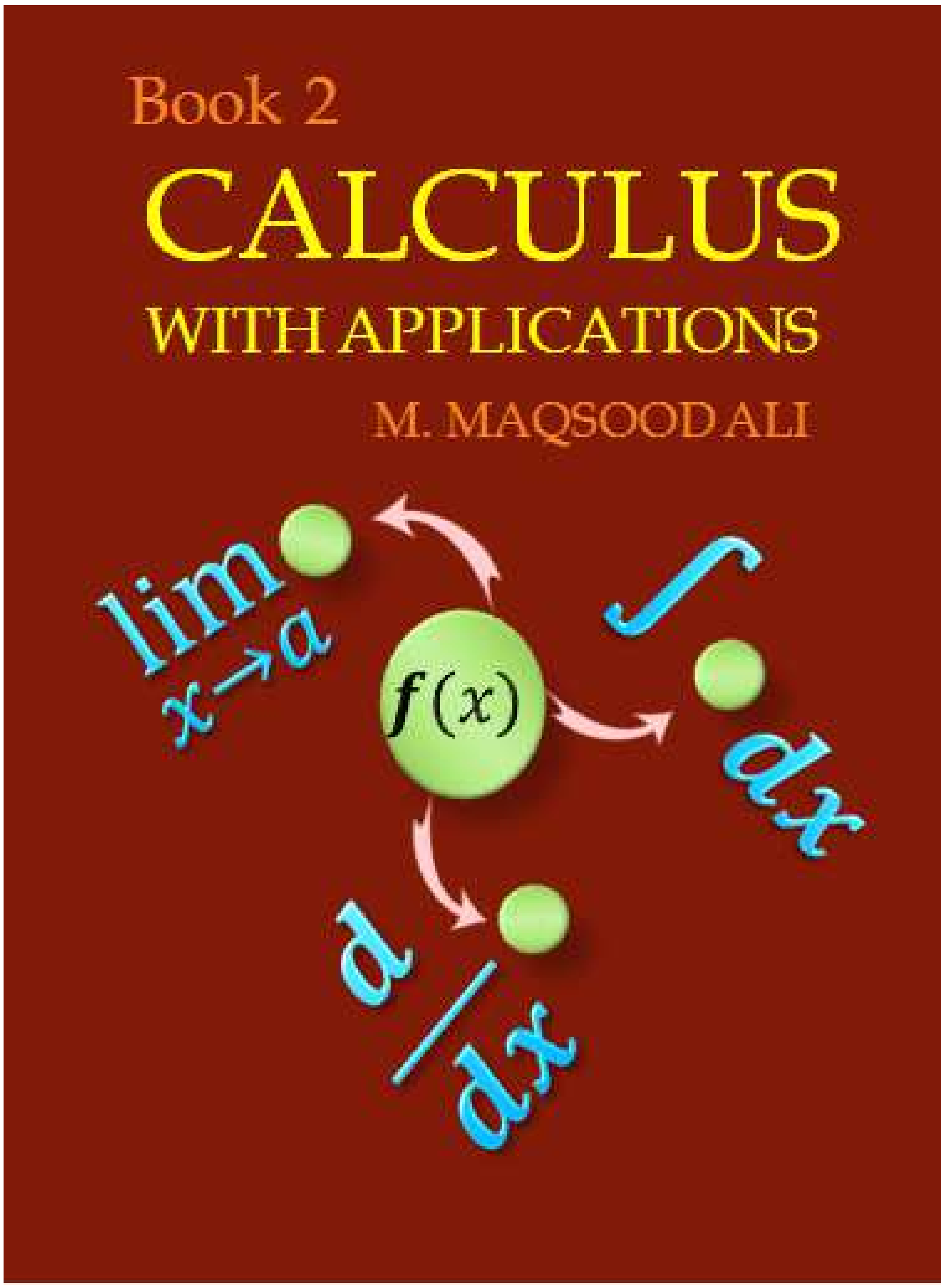

Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI

RIEMANN INTEGRAL

What will be the procedure to find the area under the curve $f(x)$ such that the function $f(x)$ is unknown?

What will be the procedure to find the area under the curve $f(x)$ without using integration, such that the function $f(x)$ is unknown?

The graph of a curve $f(x)$ is shown in **figure 10.34a**.

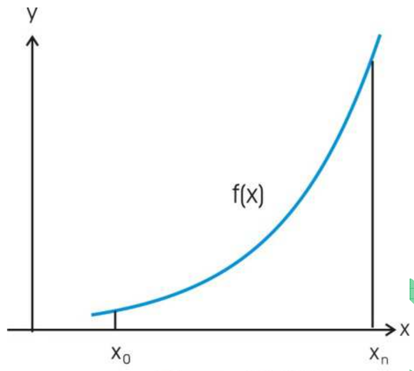


Figure 10.34a

The approximate area under the curve above x-axis between the vertical lines $x = a$ and $x = b$ can be found by dividing the interval $[a, b]$ into n subintervals , such as the region R will be divided into n strips of equal or unequal widths as shown in figure .

The accuracy of the area depends on more number of subintervals. The interval $[a, b]$ can be divided into 3, 5 and n subintervals as shown in **figure10.34 b, c, d** respectively.

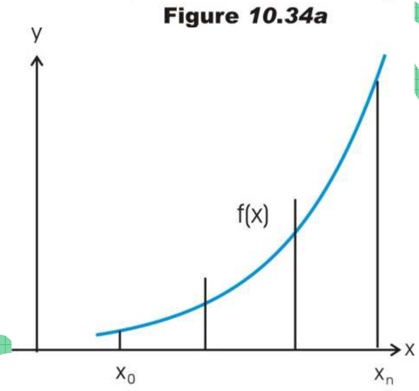


Figure 10.34b

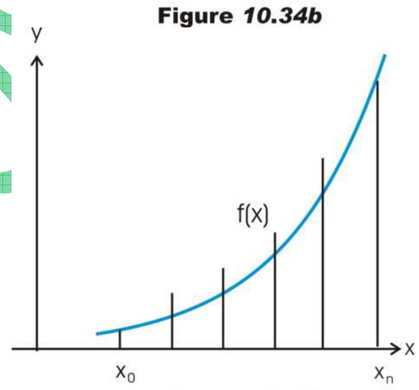


Figure 10.34c

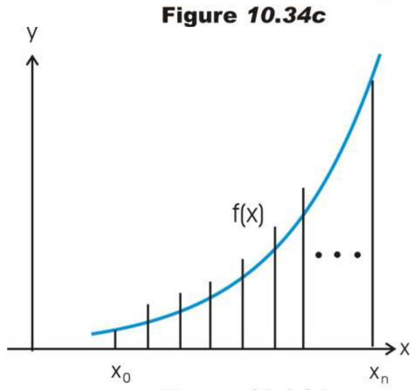


Figure 10.34d

The interval $[a, b]$ is divided into n subintervals such as

$$[x_0, x_1], [x_1, x_2], \cdots [x_{r-1}, x_r], \cdots [x_{n-1}, x_n]$$

such as

$$a = x_0 < x_1 < x_2 \cdots < x_n = b$$

where $a = x_0$ and $b = x_n$.

If the strips are of equal widths, then the widths of the strips Δx can be found as

$$\Delta x = \frac{b-a}{n}$$

If the strips are of unequal widths, then the widths of the strips $\Delta x_1, \Delta x_2, \cdots, \Delta x_r, \cdots, \Delta x_n$ can be found as

$x_1 - x_0 = \Delta x_1$, $x_2 - x_1 = \Delta x_2, \cdots, x_r - x_{r-1} = \Delta x_r, \cdots,$
$x_n - x_{n-1} = \Delta x_n$.

The area under the curve will be accurate if

$\Delta x \to 0$, for equal intervals and

$max\ \Delta x_r \to 0$, for unequal intervals.

where Δx_r : The largest width of the interval in all subintervals.

Some methods such as Lower Sum, Upper Sum, RiemannSum are given below to find the area under the curve $f(x)$, which are based on

$$Area\ of\ rectangle = length \times width$$

The interval $[a, b]$ is divided into n subintervals, so the region R under the curve is divided into n partitions of areas A_1, A_2 and $A_3, \cdots, A_n$ respectively.

1- LOWER SUM:

Let $m_r = f(p_r)$ be the greatest lower bound of $f(x)$ on $[x_{r-1}, x_r]$, $r = 1,2,3,\cdots,n$.

So $(p_r, f(p_r))$ is the absolutely minimum point on the curve on the interval $[x_{r-1}, x_r]$, $r = 1,2,3,\cdots,n$.

The horizontal lines are drawn touching or passing through these points to form the rectangles, as shown in **figure 10.35**.

The lower sum S_L is the sum of the areas of these rectangles.

$$S_L = m_1.\Delta x_1 + m_2.\Delta x_2 + m_3.\Delta x_3 + \cdots + m_n.\Delta x_n$$

or

$$S_L = f(p_1).\Delta x_1 + f(p_2).\Delta x_2 + f(p_3).\Delta x_3 + \cdots + f(p_n).\Delta x_n$$

The lower sum S_L is approximated area under the curve.

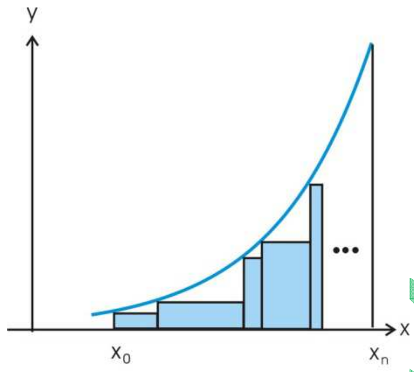


Figure 10.35

2- UPPER SUM:

Let $M_r = f(q_r)$ be the least upper bound of $f(x)$ on $[x_{r-1}, x_r]$, $r = 1,2,3,\cdots,n$.

So $(q_r, f(q_r))$ is the absolutely maximum point on the curve on the interval $[x_{r-1}, x_r]$, $r = 1,2,3,\cdots,n$.

The horizontal lines are drawn touching or passing through these points to form the rectangles, as shown in **figure 10.36**.

The upper sum S_U is sum of the areas of these rectangles.

$$S_U = M_1.\Delta x_1 + M_2.\Delta x_2 + M_3.\Delta x_3 + \cdots + M_n.\Delta x_n$$

or

$$S_U = f(q_1).\Delta x_1 + f(q_2).\Delta x_2 + f(q_3).\Delta x_3 + \cdots + f(q_n).\Delta x_n$$

The upper sum S_U is approximated area under the curve.

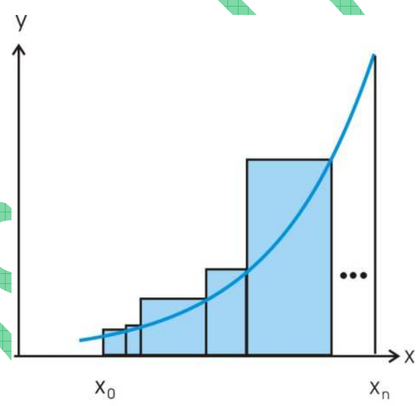


Figure 10.36

3- RIEMANN SUM:

To complete the rectangles, the horizontal lines are drawn touching or passing through the points, which may be absolutely minimum, absolutely maximum, mid-point or any suitable point on the curve, as shown in **figure 10.37**.

Let $(t_r, f(t_r))$ be the suitable point on the curve in the interval $[x_{r-1}, x_r]$, $r = 1,2,3,\cdots,n$.

The Riemann sum S_R is the sum of the areas of these rectangles.

$$S_R = f(t_1).\Delta x_1 + f(t_2).\Delta x_2 + f(t_3).\Delta x_3 + \cdots + f(t_n).\Delta x_n$$

The Riemann sum S_R is approximated area under the curve.

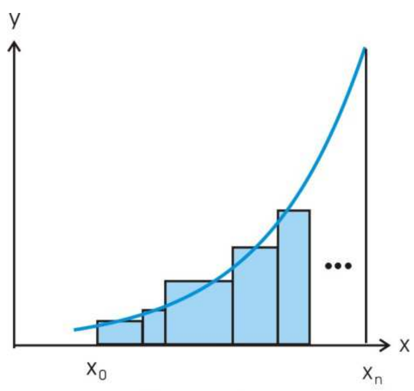


Figure 10.37

Riemann sum for n subintervals such that $n \to \infty$:

The interval $[a, b]$ is divided into n subintervals.
The Riemann sum is

$$S_R = \sum_{r=1}^{n} f(t_r).\Delta x_r$$

If $n \to \infty$ when $max\,\Delta x_r \to 0$

$$S_L = S_R = S_U$$

$$\lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(p_r).\Delta x_r = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(q_r).\Delta x_r$$

Its mean, if

$$S_L = S_U$$

$$\lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(p_r).\Delta x_r = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(q_r).\Delta x_r = L$$

then

$$S_R = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r = L$$

Riemann Integral:

If the limit $\lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r$ exists and f is defined on $[a, b]$, then f is called Riemann integrable on $[a, b]$.

Riemann Integral and Definite Integral:

If f is integrable on $[a, b]$, then definite integral from a to b is equal to Riemann integral.

$$\int_a^b f(x)dx = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r$$

Example 10.11:

$$f(x) = x^2 \quad , \quad x \in [1,6]$$

The interval [1,6] is divided into 5 subintervals, such as

$[1,2.2], [2.2,3.6], [3.6,4.2], [4.2,5.2], [5.2,6]$.

Find

(a) The lower sum of f on [1,6].

(b) The upper sum of f on [1,6].

(c) The Riemann sum of f on [1,6].

(d) The definite integral of f on [1,6].

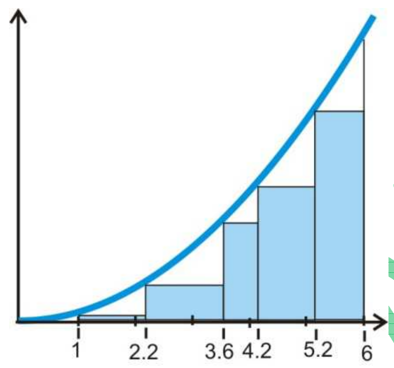
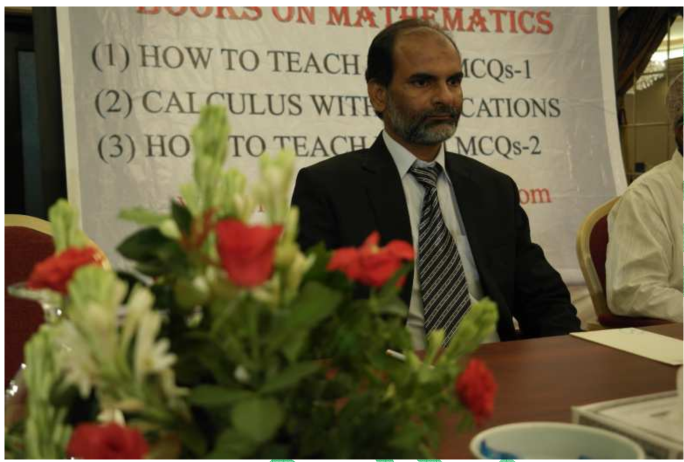


Figure *10.38*



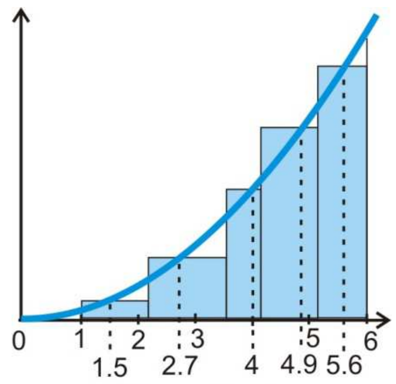


Figure *10.40*

Example 10.12:

$$f(x) = x^2 \quad , \quad x \in [1,41]$$

Divide interval [1,6] into 10 equal intervals and find

(a) The lower sum of f on [1,41].
(b) The upper sum of f on [1,41].
(c) The Riemann sum of f on [1,41].
(d) The definite integral of f on [1,41].

Solution:

$a = 1 \quad , \quad b = 41 \quad and \quad n = 10$

For 10 equal intervals, the width of the each subinterval

$$\Delta x = \frac{b-a}{n} = \frac{41-1}{10} = 4$$

The 10 subintervals are

[1,5], [5,9], [9,13], [13,17], [17,21], [21,25], [25,29], [29,33], [33,37], [37,41]

(a) The greatest lower bounds of $f(x)$ on each subintervals are

$m_1 = f(1) = 1,\ m_2 = f(5) = 25,\ m_3 = f(9) = 81,$
$m_4 = f(13) = 139,\ m_5 = f(17) = 289,\ m_6 = f(21) = 441$
$m_7 = f(25) = 625,\ m_8 = f(29) = 841,\ m_9 = f(33) = 1089$
$m_{10} = f(37) = 1369$

Figure 10.41

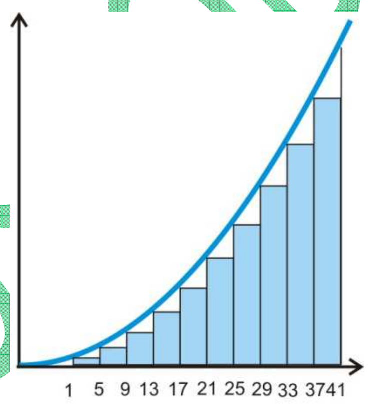


Figure 10.41

The lower sum of f on $[1,41]$ is

$$S_L = \sum_{r=1}^{10} m_r\,\Delta x_r = \sum_{r=1}^{10} m_r\,\Delta x = \left(\sum_{r=1}^{10} m_r\right)\Delta x$$

$= (1 + 25 + 81 + 139 + 289 + 441 + 625 + 841 + 1089 + 1369)4$
$= 19600$

(b) The least upper bounds of $f(x)$ on each subintervals are

$M_1 = f(5) = 25,\ M_2 = f(9) = 81,\ M_3 = f(13) = 139,$
$M_4 = f(17) = 289,\ M_5 = f(21) = 441, M_6 = f(25) = 625\ ,$
$M_7 = f(29) = 841,\ M_8 = f(33) = 1089, M_9 = f(37) = 1369$
$M_{10} = f(41) = 1681$

Figure 10.42

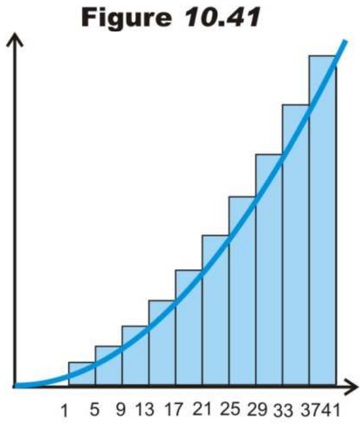


Figure 10.42

The upper sum of f on $[1,41]$ is

$$S_U = \sum_{r=1}^{10} M_r\,\Delta x_r = \sum_{r=1}^{10} M_r\,\Delta x = \left(\sum_{r=1}^{10} M_r\right)\Delta x$$

$= (\ 25 + 81 + 139 + 289 + 441 + 625 + 841 + 1089 + 1369$
$\quad +1681) \times\ 4$
$= 26320$

(c) The middle numbers on each subinterval are
3, 7, 11, 15, 19, 23, 27, 31, 35, 39
which are suitable numbers.

The values of $f(x)$ at these numbers are
$f(3) = 9, f(7) = 49, f(11) = 121, f(15) = 225, f(19) = 361$
$f(23) = 529, f(27) = 729, f(31) = 961, f(35) = 1225$
$f(39) = 1521$

Figure 10.43

Riemann sum of f on $[1,41]$ is

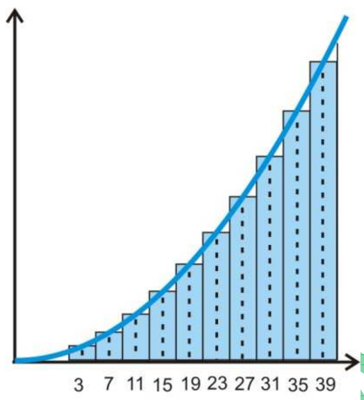


Figure ***10.43***

(a) The lower sum of f on $[0,1]$, **figure 10.44**

$$S_L = m_1 .\Delta x_1 + m_2.\Delta x_2 + m_3.\Delta x_3 + \cdots + m_n.\Delta x_n$$

$$= \frac{1}{n}\left[e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{(n-1)/n}\right]$$

$$\lim_{\max \Delta x_r \to 0} S_L = \lim_{n\to\infty} \frac{1}{n}\left[\frac{e-1}{e^{1/n}-1}\right] \quad , \quad \left\{\begin{array}{l}\text{For geometric series} \\ S_n = \dfrac{a\,(r^n - 1)}{r-1}\end{array}\right.$$

Let $\frac{1}{n} = k$

$n \to \infty \;\Rightarrow\; k \to 0$

$$\lim_{\max \Delta x_r \to 0} S_L = \lim_{k\to 0} \frac{(e-1)}{\frac{e^k - 1}{k}}$$

Since $\lim_{k\to 0} \frac{(e^k - 1)}{k} = 1$, so

$$\lim_{\max \Delta x_r \to 0} S_L = e - 1$$

(b) The upper sum of f on $[0,1]$, **figure 10.45**

$$S_U = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + \cdots + M_n\Delta x_n$$

$$= \frac{1}{n}\left[e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e\right]$$

$$= \frac{1}{n}\left[\frac{e^{\frac{1}{n}}(e-1)}{e^{1/n}-1}\right] \quad , \quad \left\{\begin{array}{l}\text{For geometric series} \\ S_n = \dfrac{a\,(r^n - 1)}{r-1}\end{array}\right.$$

$$\lim_{\max \Delta x_r \to 0} S_U = \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^{\frac{1}{n}}(e-1)}{e^{1/n}-1}\right]$$

Let $\frac{1}{n} = k$

$n \to \infty \;\Rightarrow\; k \to 0$

$$\lim_{\max \Delta x_r \to 0} S_U = \lim_{k\to 0} \frac{e^k(e-1)}{\frac{(e^k - 1)}{k}}$$

since $\lim_{k\to 0} \frac{(e^k - 1)}{k} = 1$, so

$$\lim_{\max \Delta x_r \to 0} S_U = \mathrm{e} - 1$$

(c) since

$$\lim_{\max \Delta x_r \to 0} S_L = \lim_{\max \Delta x_r \to 0} S_U = e - 1$$

So the Riemann sum of f on $[0,1]$, **figure 10.46**, is

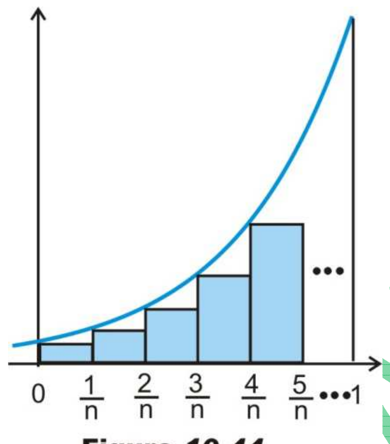


Figure 10.44

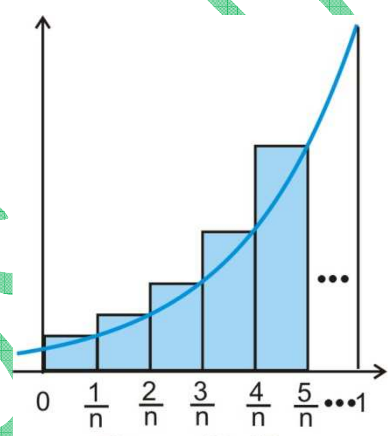


Figure 10.45

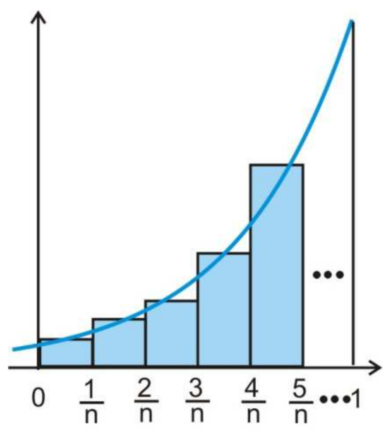


Figure 10.46

$$\lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r = e - 1$$

(d) The definite integral of f on $[0,1]$, **figure 10.47**, is

$$\int_0^1 e^x \, dx = [e^x]_0^1 = e - 1$$

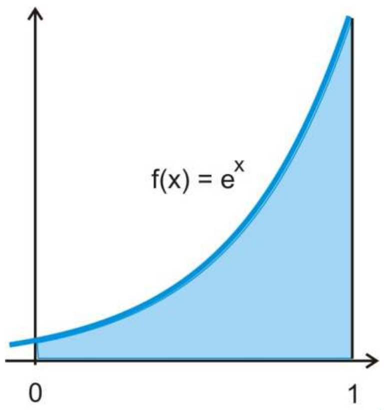


Figure 10.47

EXERCISE

(1) Divide the interval $[2,5]$ into five subintervals and find

(a) Lower sum of f on $[2,5]$.

(b) Upper sum of f on $[2,5]$.

(c) Riemann sum of f on $[2,5]$, selecting a suitable number from each subinterval.

(d) $\int_2^5 f(x)dx$

for the following

(i) $f(x) = x$

(ii) $f(x) = x^2 + 3$

(iii) $f(x) = 8$

(iv) $f(x) = x^3 + 5$

(2) Divide the interval $[0,3]$ into n subintervals, where $n \to \infty$ *and* $\max \Delta x_r \to 0$ and find

(a) Lower sum of f on $[0,3]$.

(b) Upper sum of f on $[0,3]$.

(c) Riemann sum of f on $[0,3]$.

(d) $\int_0^3 f(x)dx$

for the following

(i) $f(x) = x^3$

(ii) $f(x) = x^2$

(iii) $f(x) = e^x$

(3) Find the Riemann sum of

$$f(x) = x^3$$

on $[0, p]$, p is a positive real number, dividing the interval into n subintervals.

DEFINITE INTEGRAL:

If a function is integrable on $[a, b]$ then definite integral from a to b is defined as

$$\int_a^b f(x)dx$$

and

$$\int_a^b f(x)dx = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r$$

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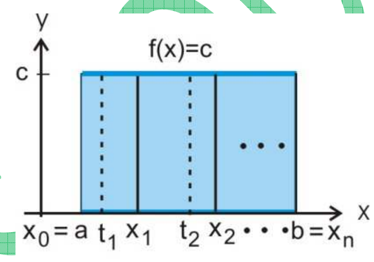

Figure 10.48

so

$$\int_a^b c\,dx = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} c.\Delta x_r$$

$$= \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} c.(\,x_r - \;x_{r-1})$$

$$= c.\lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} (\,x_r - \;x_{r-1})$$

$$= c(b-a)$$

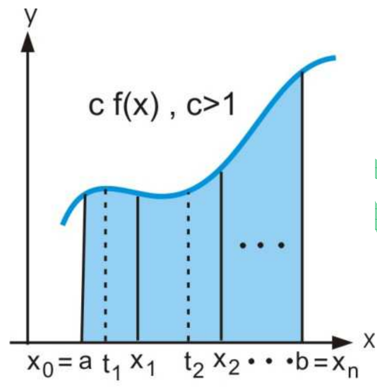

Figure 10.49a

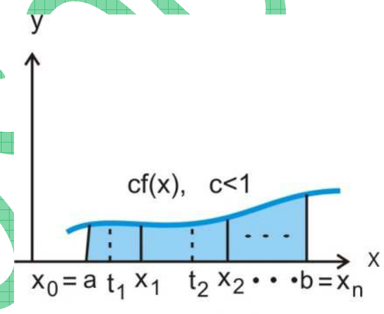

Figure 10.49b

Proof (ii):

$$\int_a^b c\,f(x)dx = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} c\,f(t_r).\Delta x_r$$

$$= c\lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r$$

$$= c\int_a^b f(x)dx$$

Figure 10.49 a, b and 10.50

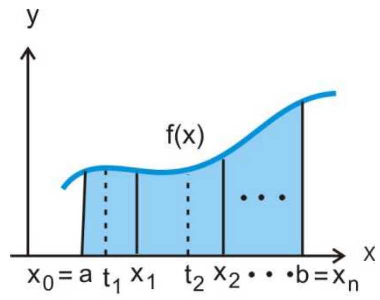

Figure 10.50

Proof (iii):

since $\displaystyle\sum_{r=1}^{n} f(t_r).\Delta x_r + \sum_{r=1}^{n} g(t_r).\Delta x_r$

$$= \{f(t_1).\Delta x_1 + f(t_2).\Delta x_2 + \cdots + f(t_n).\Delta x_n\}$$
$$+\{g(t_1).\Delta x_1 + g(t_2).\Delta x_2 + \cdots + g(t_n).\Delta x_n\}$$

$$= \{\,f(t_1) + g(t_1)\}\Delta x_1 + \{f(t_2) + g(t_2)\}\Delta x_2$$
$$+ \cdots + \{f(t_n) + g(t_n)\}\Delta x_n$$

$$= \sum_{r=1}^{n} \{f(t_r) + g(t_r)\}.\Delta x_r$$

Now

$$\int_a^b f(x)dx + \int_a^b g(x)dx$$

$$= \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f(t_r).\Delta x_r + \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} g(t_r).\Delta x_r$$

$$= \lim_{\max \Delta x_r \to 0} \left\{ \sum_{r=1}^{n} f(t_r).\Delta x_r + \sum_{r=1}^{n} g(t_r).\Delta x_r \right\}$$

$$= \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} \{f(t_r) + g(t_r)\}.\Delta x_r$$

$$= \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} (f+g)(t_r).\Delta x_r$$

$$= \int_a^b (f+g)(x)dx$$

$$= \int_a^b [f(x) + g(x)]dx$$

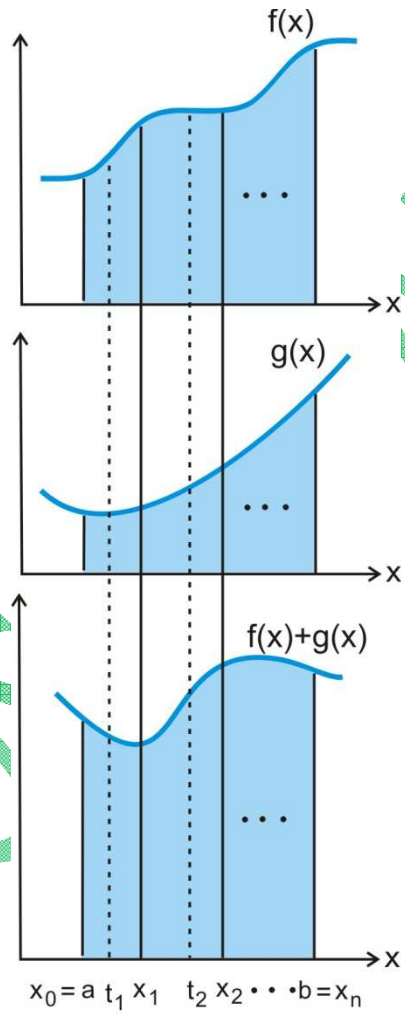

Figure 10.51 a,b,c

Figure 10.51 a, b, c

(iv) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \,, c \in [a,b]$

FUNDAMENTAL THEOREM:

If f is integrable on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Proof:

By Lagrange's mean value theorem

$$f'(t_r) = \frac{f(x_r) - f(x_{r-1})}{x_r - x_{r-1}}$$

$$f(x_r) - f(x_{r-1}) = f'(t_r)(x_r - x_{r-1})$$

Putting the values of r from 0 to n

$$f(x_1) - f(x_0) = f'(t_1)(x_1 - x_0) = f'(t_1).\Delta x_1$$

$$f(x_2) - f(x_1) = f'(t_2)(x_2 - x_1) = f'(t_2).\Delta x_2$$

$$f(x_3) - f(x_2) = f'(t_3)(x_3 - x_2) = f'(t_3).\Delta x_3$$

$$\vdots$$

$$f(x_n) - f(x_{n-1}) = f'(t_n)(x_n - x_{n-1}) = f'(t_n).\Delta x_n$$

By adding

$$f(x_n) - f(x_0) = f'(t_1).\Delta x_1 + f'(t_2).\Delta x_2 + \cdots + f'(t_n).\Delta x_n$$

$$f(x_n) - f(x_0) = \sum_{r=1}^{n} f'(t_r).\Delta x_r$$

Putting $x_0 = a, x_n = b$ *and* $\max \Delta x_r \to 0$

$$f(b) - f(a) = \lim_{\max \Delta x_r \to 0} \sum_{r=1}^{n} f'(t_r).\Delta x_r$$

$$f(b) - f(a) = \int_a^b f'(x)dx$$

$$\int_a^b f'(x)dx = f(b) - f(a)$$

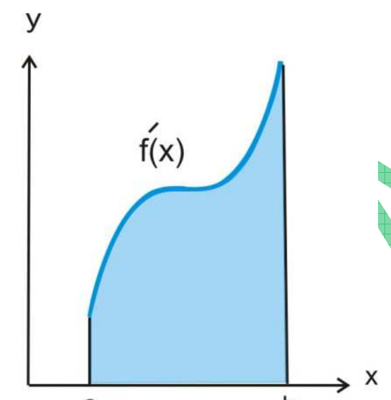

Figure *10.52* a

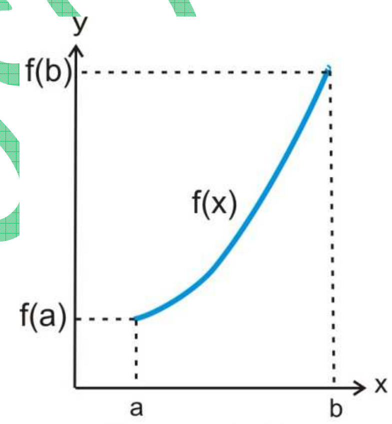

Figure *10.52* b

Figure 10.52 a, b