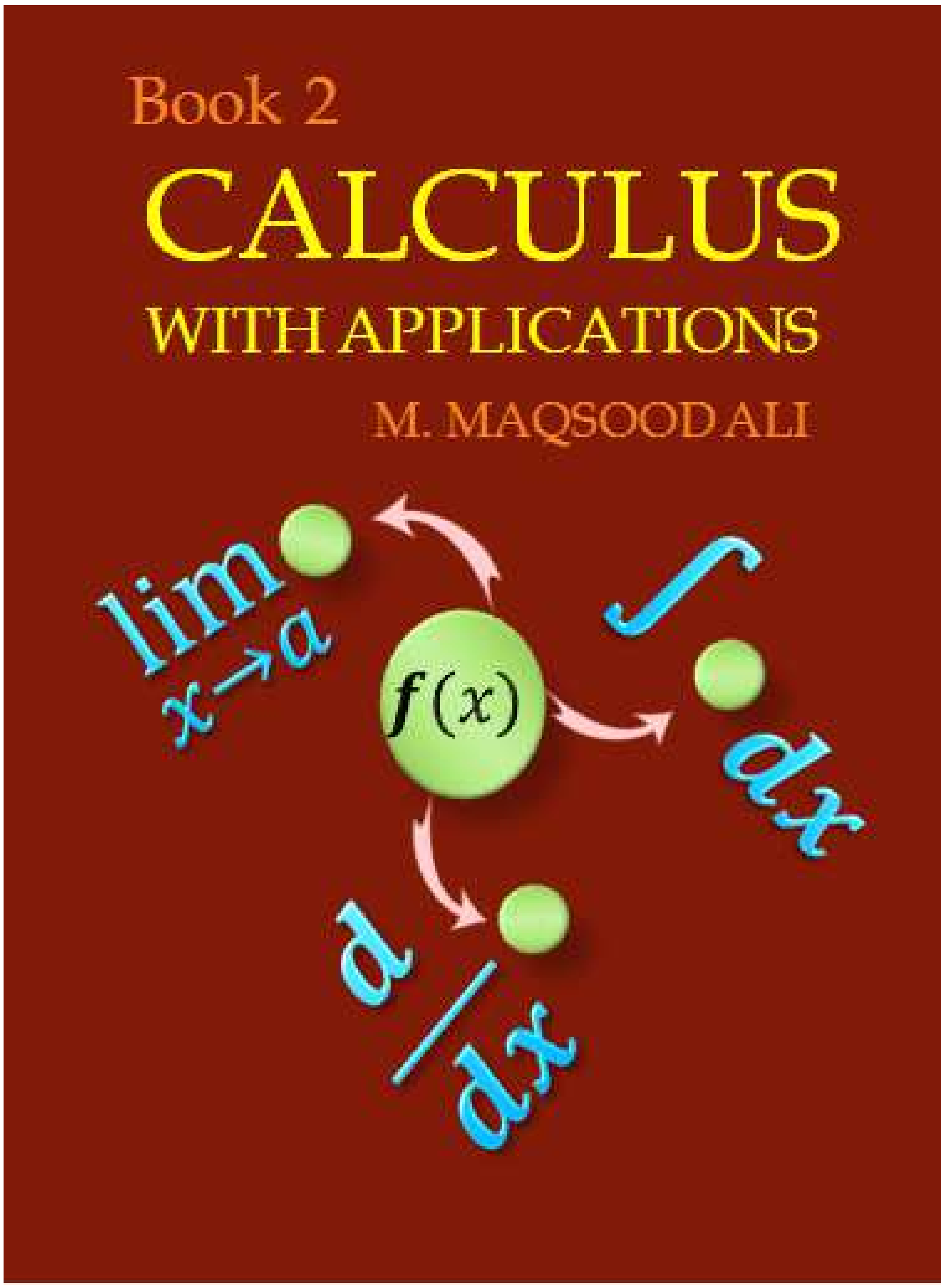
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
d/dx

REDUCTION FORMULAE
(Trigonometric Function)

$(i)\ \int sin^n\, x\, dx$

$$G_n = \int sin^n\, x\, dx$$

$$= \int sin^{n-1}\, x \sin x\, dx$$

$$G_n = sin^{n-1}x\,(-\cos x) - \int (n-1) sin^{n-2} x cosx\,(-cosx)dx$$

$$= -sin^{n-1}x\ \cos x + (n-1)\int (n-1) sin^{n-2} x cos^2 x\, dx$$

$$= -sin^{n-1}x\ \cos x + (n-1)\int (n-1) sin^{n-2} x(1 - sin^2 x)\, dx$$

$$= -sin^{n-1}x\ \cos x + (n-1)\int sin^{n-2} x\, dx - (n-1)\int sin^n\, x\, dx$$

$$= -sin^{n-1}x\ \cos x + (n-1)\ \ G_{n-2} - (n-1)\, G_n$$

$$G_n + (n-1)G_n = -sin^{n-1}x\ \cos x + (n-1)\ \ G_{n-2}$$

$$n\, G_n = -sin^{n-1}x\ \cos x + (n-1)\ \ G_{n-2}$$

$$G_n = -\frac{1}{n}\ sin^{n-1}\, x \cos x + \frac{(n-1)}{n} G_{n-2}$$

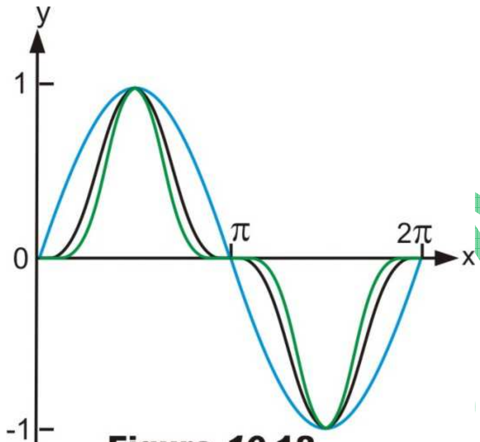

Figure 10.18

Curves sinx, sin^3x, sin^5x

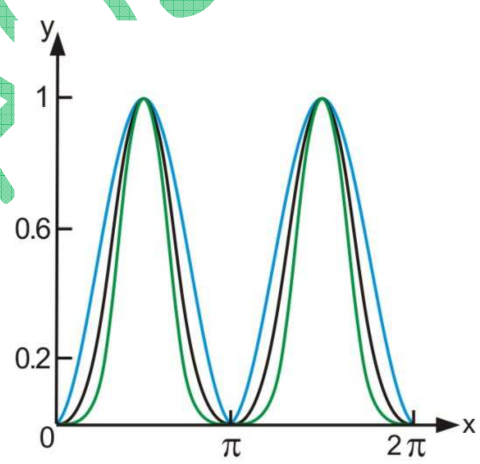

Figure 10.19

Curves sin^2x, sin^4x, sin^6x

$(ii) \int cos^n\, x\, dx$

$G_n = \int cos^n\, x\, dx$

$G_n = \int cos^{n-1}\, x \cos x\, dx$

$= cos^{n-1}\, x \sin x + (n-1) \int cos^{n-2}\, x\, sin^2\, x\, dx$

$= cos^{n-1}\, x \sin x$
$+ (n-1) \int cos^{n-2}\, x\, (1 - cos^2\, x)\, dx$

$= cos^{n-1}\, x \sin x$
$+ (n-1) \int cos^{n-2}\, x\, dx - (n-1) \int cos^n\, x\, dx$

$= cos^{n-1}\, x \sin x + (n-1)\, G^{n-2} - (n-1) G_{n-2}$

$G_n + (n-1)\, G_n = cos^{n-1}\, x \sin x + (n-1)\, G_{n-2}$

$nG_n = cos^{n-1}\, x \sin x + (n-1)\, G^{n-2}$

$G_n = \frac{1}{n} cos^{n-1}\, x \sin x + \frac{n-1}{n} G_{n-2}$

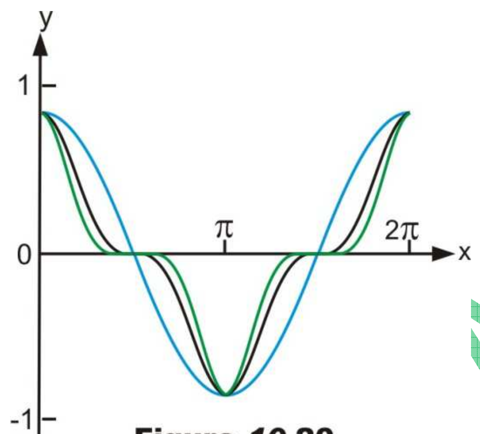

Figure 10.20

***Curves* $cosx, cos^3x, cos^5x$**

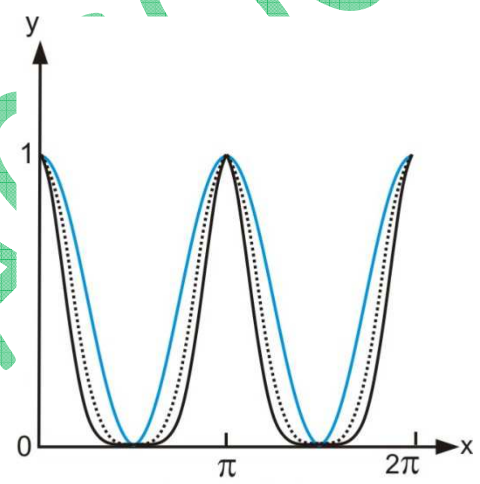

Figure 10.21

***Curves* cos^2x, cos^4x, cos^6x**

(iii) $\int tan^n\, x\, dx$

$$G_n = \int tan^n\, x\, dx$$

$$= \int tan^{n-2}\, x\, tan^{\,n}\, x\, dx$$

$$= \int tan^{n-2}\, x (sec^{\,n}\, x - 1)\, dx$$

$$= \int tan^{n-2}\, x\, sec^{\,n}\, x\, dx - \int tan^{n-2}\, x\, dx$$

$$= \frac{tan^{n-2}\, x}{n-1} - G_{n-2}$$

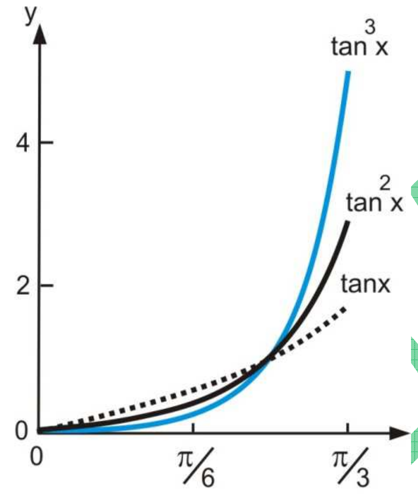

Figure 10.22

(iv) $\int sec^n\, x\, dx$

$$G_n = \int sec^n\, x\, dx$$

$$= \int sec^{n-2}\, x\, sec^2\, x\, dx$$

$$= sec^{n-2}\, x (tanx) - \int (n-2)\, sec^{n-3}\, x \sec x \tan x\ (\tan x)\, dx$$

$$= sec^{n-2}\, x \tan x - (n-2) \int sec^{n-2}\, x\, tan^2 x\, dx$$

$$= sec^{n-2}\, x \tan x - (n-2) \int sec^{n-2}\, x\, (sec^2 x - 1)\, d$$

$$= sec^{n-2}\, x \tan x - (n-2) \int sec^n\, x\, dx + (n-2) \int sec^{n-2}\, x\, dx$$

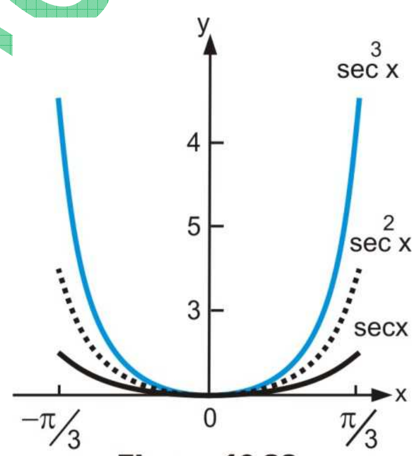

Figure 10.23

$$= sec^{n-2}\, x \tan x - (n-2)\, G_n + (n-2)\, G_{n-2}$$

$$G_n + (n-2) = sec^{n-2}\, x \tan x + (n-2)\, G_{n-2}$$

$$G_n = \frac{1}{n-1}\, sec^{n-2}\, x\, tan\, x + \frac{(n-2)}{n-1}\, G_{n-2}$$

$(v)\ Gm, n = \int \sin^m x \cos^n x\, dx$

(a) $\underline{To\ reduce\ n:}$

$$\left(1+\frac{n-1}{m+1}\right)G_{m,n} = \frac{1}{m+1}sin^{m+1}x\ cos^{n-1}x + \frac{n-1}{m+1}G_{m,n-2}$$

$$\left(\frac{m+n}{m+1}\right)G_{m,n} = \frac{1}{m+1}sin^{m+1}x\ cos^{n-1}x + \frac{n-1}{m+1}G_{m,n-2}$$

$$G_{m,n} = \frac{1}{m+1}sin^{m+1}x\ cos^{n-1}x + \frac{n-1}{m+1}G_{m,n-2}$$

(b) To reduce m:

$$G_{m,n} = \int sin^m x\ cos^n x\,dx$$

$$G_{m,n} = \int (cos^n x \sin x)\ sin^{m-1}x\,dx$$

$$-G_{m,n} = \int (-cos^n x \sin x)\ sin^{m-1}x\,dx$$

Integrate by parts taking $sin^{m-1}x$ as first function and $(-cos^n x \sin x)$ as second function.

$$-G_{m,n} = \int\ sin^{m-1}x\,\frac{cos^{n+1}x}{n+1}$$

$$-\int (m-1)\,sin^{m-2}\ x\,dx$$

$$= \frac{1}{n+1}\ sin^{m-2}x\ cos^{n-1}x$$

$$-\frac{m-1}{n+1}\int sin^{m-2}x\ cos^{n+2}x\,dx$$

$$-G_{m,n} = \frac{1}{n+1}\ sin^{m-2}x\ cos^{n-1}x$$

$$-\frac{m-1}{n+1}\int sin^{m-2}x\ cos^n x\ cos^2 x\,dx$$

$$-G_{m,n} = \frac{1}{n+1}\ sin^{m-2}x\ cos^{n-1}x$$

$$-\frac{m-1}{n+1}\int sin^{m-2}x\ cos^n x\,dx$$

$$+\frac{m+1}{n+1}\int sin^m x\ cos^n x\,dx$$

$$-G_{m,n} = \frac{1}{n+1}\ sin^{m-2}\ x\ \ cos^{n-1}\ x$$
$$-\frac{m-1}{n+1}\ G_{m-2,n} + \frac{m-1}{n+1}\ G_{m,n}$$
$$\frac{m-1}{n+1}\ G_{m,n} + G_{m,n} = -\frac{1}{n+1}\ sin^{m-1}\ x\ \ cos^{n+1}\ x$$
$$+\frac{m-1}{n+1}\ G_{m-2,n}$$
$$\left(\frac{m-1}{n+1}+1\right) G_{m,n} = \frac{1}{n+1} sin^{m-1}\ x\ \ cos^{n+1}\ x$$
$$+\frac{m-1}{n+1}\ G_{m-2,n}$$
$$\left(\frac{m-1}{n+1}\right) G_{m,n} = \frac{1}{n+1} sin^{m-1}\ x\ \ cos^{n+1}\ x$$
$$+\frac{m-1}{n+1}\ G_{m-2,n}$$

Life Sience

$$G_{m,n} = -\frac{1}{m+n}\ sin^{m-1}\ x\ \ cos^{n+1}\ x\ + \frac{m-1}{n+1}\ G_{m-2,n}$$

Example 10.6:

In a city, the number of cars crossing a traffic signal at time t (in hours) is modeled by a function $N(t)$. The rate at which the cars cross the signal is

$$\frac{d}{dt}N(t) = 50\ sin^4(t/7.64)$$

Figure 10.24

(a) Find the function $N(t)$, if not any car crosses the signal at $t = 0$ (01:00 a.m.).

(b) How many cars will cross the signal first day?

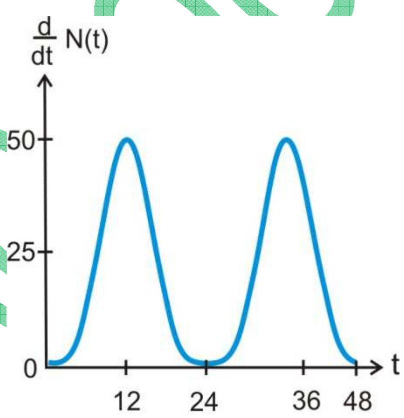

Figure 10.24

Solution:

$$\frac{d}{dt}N(t) = 50\ sin^4(t/7.64)$$
$$N(t) = 50\int sin^4(t/7.64)\ dt$$
$$Let\ \ \frac{t}{7.64} = x\ , t = 7.64x\ \Rightarrow dt = 7.64dx, so$$
$$N(x) = 382\int sin^4 x\ dx\quad \rightarrow (1)$$

Let

$$G_4 = \int sin^4\ x\ dx$$

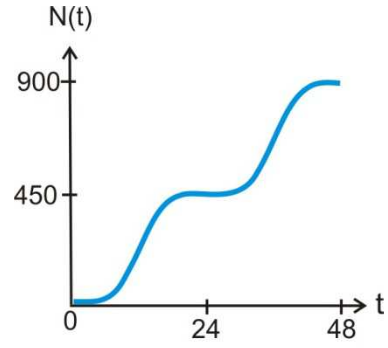

Figure 10.25

Reduction formula

$$G_n = \int sin^n x\, dx$$
$$= -\frac{1}{n} sin^{n-1} x \cos x + \frac{n-1}{n} G_{n-2}$$

$$G_4 = -\frac{1}{4} sin^3 x \cos x + \frac{3}{4} G_2$$
$$= -\frac{1}{4} sin^3 x \cos x + \frac{3}{4} \{-\frac{1}{2} \sin x \cos\ x + \frac{1}{2} G_0\}$$

$$= -\frac{1}{4} sin^3 x \cos x - \frac{3}{8} \sin x \cos\ x + \frac{3}{8} G_0$$

$$= -\frac{1}{4} sin^3 x \cos x - \frac{3}{8} \sin x \cos\ x + \frac{3}{8} \int sin^0 x\, dx$$

$$= -\frac{1}{4} sin^3 x \cos x - \frac{3}{8} \sin x \cos\ x + \frac{3}{8} x$$

So that

$$N(x) = 382 \left\{-\frac{1}{4} sin^3 x \cos x - \frac{3}{8} \sin x \cos\ x + \frac{3}{8} x\right\} + C$$

Total number of cars cross the signal at time t is

$$N(t) = 382 \left\{-\frac{1}{4} sin^3 \left(\frac{t}{7.64}\right) \cos\left(\frac{t}{7.64}\right) - \frac{3}{8} \sin\left(\frac{t}{7.64}\right) \cos\left(\frac{t}{7.64}\right) + \frac{3}{8}\left(\frac{t}{7.64}\right)\right\} + C \quad \rightarrow (2)$$

Putting $t = 0$ and $N(t) = 0$ in (2), we get

$$C = 0$$

So that

$$N(t) = 382 \left\{-\frac{1}{4} sin^3 \left(\frac{t}{7.64}\right) \cos\left(\frac{t}{7.64}\right) - \frac{3}{8} \sin\left(\frac{t}{7.64}\right) \cos\left(\frac{t}{7.64}\right) + \frac{3}{8}\left(\frac{t}{7.64}\right)\right\}$$

Figure 10.25

(b) Putting $t = 1$ day $= 24$ hours

$$N(24) = 450$$

First day total 450 cars will cross the signal.

Example 10.7: **Physical Sience**

The rate of change in temperature of a part of a electronic device in working is

$$\frac{dT}{dt} = 25cos^5 t \text{ degree celcius per minute.}$$

Figure 10.26

Find the temperature of the part of the device as a function of time. The temperature of it was 30 C^o when the experiment started.

(a) Find the rate of change in temperature at $t =$`2.5 minutes.

(b) Find the temperature of the part at $t = 2.5$ minutes.

Solution:

Reduction formula

$$G_n = \int cos^n xdx = \frac{1}{2}cos^{n-1}xsinx + \frac{n-1}{n}G_{n-2}$$

So that

$$G_5 = 25\int cos^5 xdx$$

$$= 25(\frac{1}{5}cos^4 xsinx + \frac{4}{5}G_3)$$

$$= 25\{\frac{1}{5}cos^4 xsinx + \frac{4}{5}\left(\frac{1}{3}cos^2 xsinx + \frac{2}{3}G_1\right)\}$$

$$= 25(\frac{1}{5}cos^4 xsinx + \frac{4}{15}cos^2 xsinx + \frac{8}{15}G_1)$$

$$= 25(cos^4 xsinx + \frac{4}{15}cos^2 xsinx + \frac{8}{15}\int cosxdx)$$

$$= 25(\frac{1}{5}cos^4 xsinx + \frac{4}{15}cos^2 xsinx + \frac{8}{15}sinx) + C$$

$$= 5(cos^4 xsinx + \frac{4}{3}cos^2 xsinx + \frac{8}{3}sinx) + C$$

So that

$$T(t) = 5(cos^4 tsint + \frac{4}{3}cos^2 tsint + \frac{8}{3}sint) + C$$

At $t = 0$, $T = 30$, so

$$30 = 25(0 + 0 + 0) + C$$

$$C = 30$$

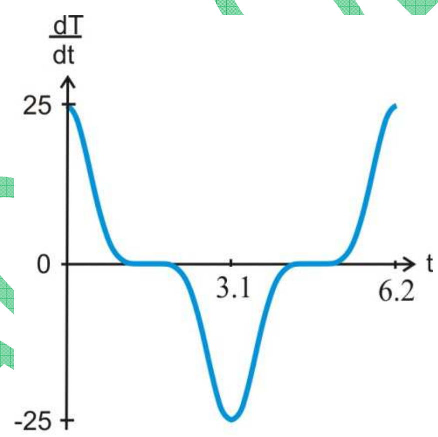

Figure 10.26

The temperature T as a function of time t is

$$T(t) = 5(cos^4 t sint + \frac{4}{3}cos^2 t sint + \frac{8}{3}sint) + 30$$

Figure 10.27

(a) Rate of change in T at $t = 2.5$ minutes.

$$\left(\frac{dT}{dt}\right)_{t=2.5} = 25cos^5 2.5$$
$$= -8.2\ {}^oC/minute$$

Negative sign shows that the temperature is decreasing at $t = 2.5$.

(b) The temperature of the part at $t = 2.5$.

$$T(2.5) = 5(cos^4 2.5\ sin2.5 + \frac{4}{3}cos^2 2.5\ sin2.5$$
$$+\frac{8}{3}sin2.5) + 30$$
$$= 32.25\ {}^oC$$

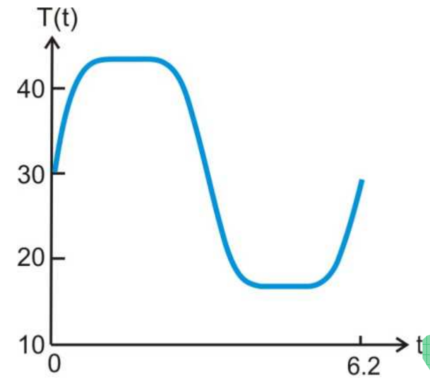

Figure *10.27*

Example 10.8: **Life Sience**

Far from the city a oil tanker hits a reef and begins to leak. A team reaches there from the city to repair the leak. They note that 200 liters oil has been leaked and the remaining oil is leaking at the following rate

$$\frac{d}{dt}V(t) = 10\ tan^6\left(\frac{t+10}{20}\right)$$

Figure 10.28

where t is the time (in hours) and $V(t)$ is the total volume (in hundred of liters) of leaked oil at time t.

(a) Find the total volume of leaked oil at time t.

(b) How much oil will have leaked altogather at the end of 4 hours?

(c) How much oil will have leaked altogather at the end of 10 hours?

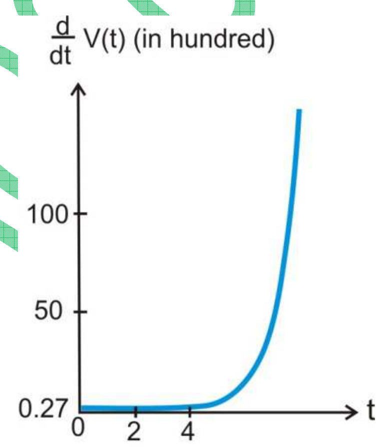

Figure *10.28*

Solution:

$$\frac{d}{dt}V(t) = 10\ tan^6\left(\frac{t+10}{20}\right)$$

$$V(t) = \int 10\ tan^6\left(\frac{t+10}{20}\right)dt \quad \rightarrow (1)$$

Let $\frac{t+10}{20} = x\ ,\ \ t = 20x - 10\ \Rightarrow dt = 20dx$

Putting in (1)

$$V(x) = 200\int tan^6 x\, dx \quad \rightarrow (2)$$

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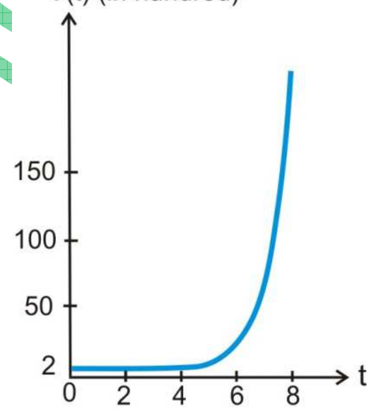

Figure 10.29

Reduction formulae for logarithmic function:

$$G_n = \int (In\, x)^n\, dx$$

It can be written as

$G_n = \int 1\,.\,(In\, x)^n\, dx$

Integrate by parts taking 1 as second function:

$$G_n = (In\, x)^n\,(x) - \int n\,(In\, x)^{n-1}\left(\frac{1}{x}\right)(x)\, dx$$

$$G_n = x\,(In\, x)^n - n\int (In\, x)^{n-1}\, dx$$

$G_n = x\,(In\, x)^n - n\, G_{n-1}$

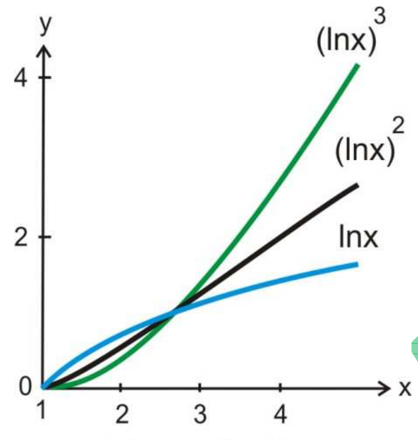

Figure 10.30

Example 10.9:

The marginal profit from the sale of shoes is

$$\overline{MP}(q) = \{\ln(q+3)\}^5$$

Figure 10.31

Business and Economics

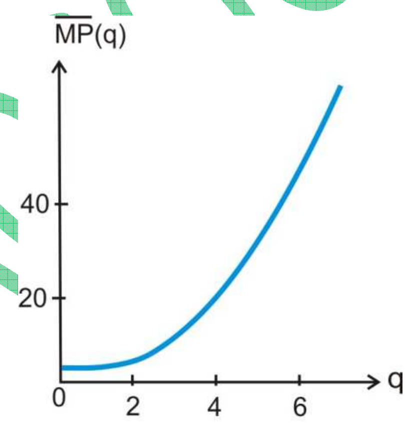

Figure 10.31

where q is the number of pairs of shoes (in hundreds) and P is the profit (in hundred of dollars).

(a) Find the profit function $P(q)$, if the loss is \$3000 for no sale of shoes.

(b) Calculate the loss on the sale of 200 pairs of shoes.

(c) Calculate the profit on the sale of 600 pairs of shoes.

Solution:

$$\overline{MP}(q) = \{\ln(q+3)\}^5$$

$$P(q) = \int \{\ln(q+3)\}^5 dq$$

Let $q+3 = x \Rightarrow dq = dx$

$$P(x) = \int \{\ln x\}^5 dx$$

$$If\ G_n = \int (In\, x)^n\, dx\ then$$

Reduction formula

$$G_n = x\,(In\, x)^n - n\, G_{n-1}$$

$G_5 = x\,(In\, x)^5 - 5G_4$

$= x\,(In\, x)^5 - 5\,\{x\,(In\, x)^4 - 4G_3\}$

$= x\,(In\, x)^5 - 5x\,(In\, x)^4 + 20G_3$

$= x\,(In\, x)^5 - 5x\,(In\, x)^4 + 20\{x\,(In\, x)^3 - 3G_2\}$

$= x\,(In\, x)^5 - 5x\,(In\, x)^4 + 20x\,(In\, x)^3 - 60G_2$

$G_5 = x\,(In\,x)^5 - 5x\,(In\,x)^4 + 20x\,(In\,x)^3$
$\quad - 60\{x\,(In\,x)^2 - 2G_1\}$

$G_5 = x\,(In\,x)^5 - 5x\,(In\,x)^4 + 20x\,(In\,x)^3$
$\quad - 60x\,(In\,x)^2 + 120G_1$

$G_5 = x\,(In\,x)^5 - 5x\,(In\,x)^4 + 20x\,(In\,x)^3$
$\quad - 60x\,(In\,x)^2 + 120\,(x\,In\,x - G_0)$

$G_5 = x\,(In\,x)^5 - 5x\,(In\,x)^4 + 20x\,(In\,x)^3$
$\quad - 60x\,(In\,x)^2 + 120x\,In - 120G_0$

$G_5 = x\,(In\,x)^5 - 5x\,(In\,x)^4 + 20x\,(In\,x)^3$
$\quad - 60x\,(In\,x)^2 + 120x\,Inx - 120x$

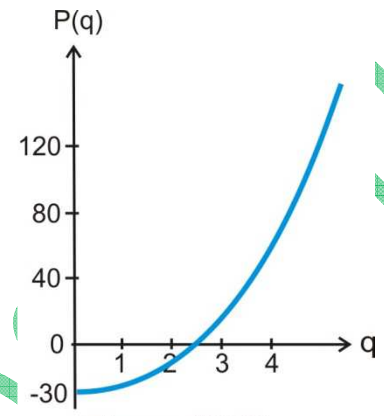

Figure 10.32

$G_5 = x\,\{(In\,x)^5 - 5\,(In\,x)^4 + 20\,(In\,x)^3$
$\quad - 60\,(In\,x)^2 + 120\,Inx - 120\} + C$

So that

$P(x) = x\,\{(In\,x)^5 - 5\,(In\,x)^4 + 20\,(In\,x)^3$
$\quad - 60\,(In\,x)^2 + 120\,Inx - 120\} + C \;\rightarrow (1)$

Putting $x = q + 3$ in (1)

$P(q) = (q+3)\,\{(In\,(q+3))^5 - 5\,(In\,(q+3))^4$
$\quad + 20\,(In\,(q+3))^3 - 60(\ln(q+3))^2$
$\quad +120ln(q+3) - 120\} + C \qquad \rightarrow (2)$

Loss is \$3000 for no sale of shoes.

Putting $P(x) = -30$ and $x = 0$ in (2)

$$C = 90$$

Putting in (2)

$P(q) = (q+3)\,\{(ln\,(q+3))^5 - 5\,(ln\,(q+3))^4$
$\quad + 20\,(ln\,(q+3))^3 - 60(\ln(q+3))^2$
$\quad +120ln(q+3) - 120\} + 90 \qquad \rightarrow (3)$

is the profit function.

Figure 10.32

(b) For 200 shoes ,Putting $x = 2$ in (3)

$$P(2) = -18.28$$

The loss is \$1828.

(c) For 600 shoes ,Putting $x = 6$ in (3)

$$P(6) = 97.45$$

The profit is \$9745.

Reduction formulae which will reduce n:

$$G_{m,n} = \int x^m \, (Inx)^n \, dx$$

Integrate by parts

$$G_{m,n} = (Inx)^n \frac{x^{m+1}}{m+1} - \int n \, (Inx)^{n-1} \left(\frac{1}{x}\right) \left(\frac{x^{m+1}}{m+1}\right) dx$$

$$G_{m,n} = \frac{x^{m+1}}{m+1} (Inx)^n - \frac{n}{m+1} \int x^m \, (Inx)^{n-1} \, dx$$

$$G_{m,n} = \frac{x^{m+1}(Inx)^n}{m+1} - \frac{n}{m+1} G_{m,n-1}$$

Example 10.10: Life Sience

The number of trees burning at time t (in days) in a forest fire is modeled by a function $N(t)$. The fire is spread at a rate

$$\frac{dN}{dt} = (t+2)^3\{\ln (t+2)\}^3$$

Figure 10.33

(a) Find the function $N(t)$, if 50 trees are burning originally.

(b) How many trees will be on fire 3 days later?

Solution:

$$\frac{dN}{dt} = (t+2)^3\{\ln (t+2)\}^3$$

Figure 10.33

$$N(t) = \int (t+2)^3\{\ln (t+2)\}^3 \, dt \quad \rightarrow (1)$$

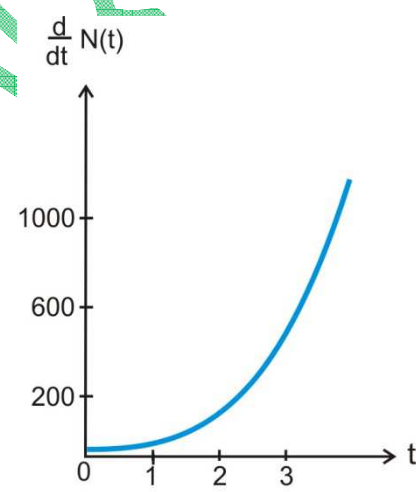

Figure 10.33

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Putting in (3)

$$N(t) = \frac{(t+2)^4}{4}\left[\{In(t+2)\}^3 - \frac{3}{4}\{\ln(t+2)\}^2 + \frac{3}{8}\ln(t+2) - \frac{3}{8}\right] + 50 \quad \rightarrow (3)$$

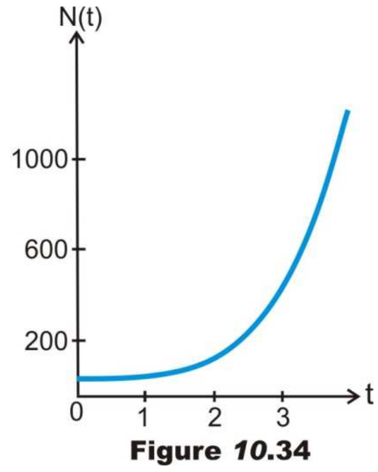

Figure 10.34

(b) Putting $t = 3$ days in (3)

$$N(3) = 433$$

433 trees will on fire after 3 days.

EXERCISE

(1) $\int cosec^n x\, dx$
(2) $\int cot^n x\, dx$
(3) $\int sin^5 x\, dx$
(4) $\int cos^6 x\, dx$
(5) $\int tan^4 x\, dx$
(6) $\int sec^7 x\, dx$
(7) $\int cosec^6 x\, dx$
(8) $\int cot^5 x\, dx$
(9) $\int sin^5 x\, cos^5\, x\, dx$ reduce power of $\sin x$.
(10) $\int sin^2 x\, cos^4\, x\, dx$ reduce power of $\cos x$.
(11) $\int (In\, x)^5\, dx$
(12) $\int x^4\, (In\, x)^3\, dx$