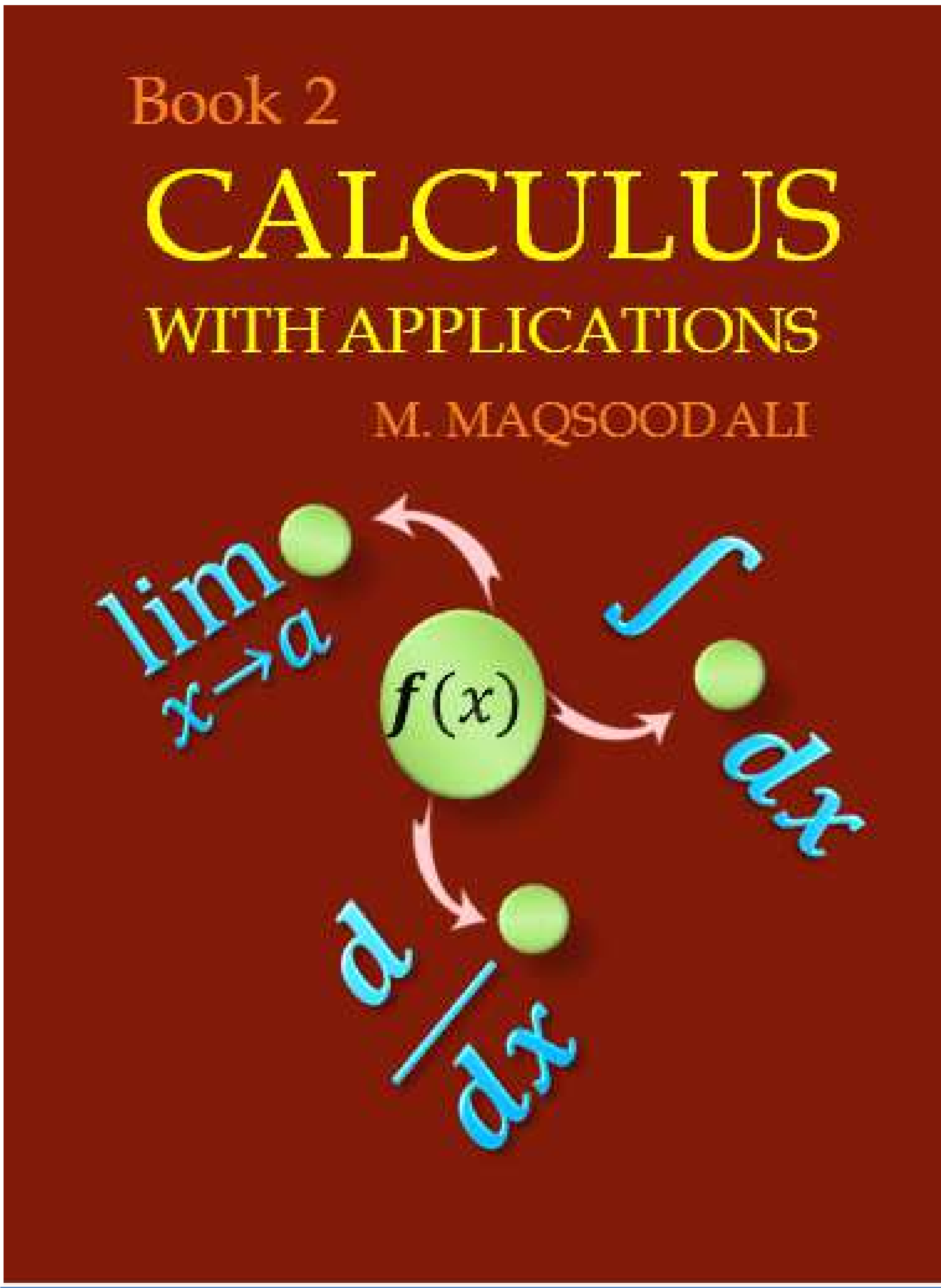
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
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REDUCTION FORMULAE

A reduction formula is formed for $\int f^n(x)dx$ or $\int f^n(x)g^m(x)dx$ when the integration is completed repeating the same process more than two times and each time except the last time $f(x)$ and $g(x)$ remain same only the powers m or n reduce.

The following four reduction formulae shows the reduction the power of x or $(a+bx^n)$.

(1) $\int x^m(a+bx^n)^r dx$

Reduction formula to reduce the power of x

$$\int x^m(a+bx^n)^r dx = Pf(x) + Q\int x^{m-n}(a+bx^n)^r dx$$

(2) $\int x^{-m}(a+bx^n)^r dx$

Reduction formula to reduce the power of x

$$\int x^{-m}(a+bx^n)^r dx = Pf(x) + Q\int x^{-m+n}(a+bx^n)^r dx$$

(3) $\int x^m(a+bx^n)^r dx$

Reduction formula to reduce the power of $(a+bx^n)$.

$$\int x^m(a+bx^n)^r dx = Pf(x) + Q\int x^m(a+bx^n)^{r-1} dx$$

(4) $\int x^m(a+bx^n)^{-r} dx$

Reduction formula to reduce the power of $(a+bx^n)$.

$$\int x^m(a+bx^n)^{-r} dx = Pf(x) + Q\int x^m(a+bx^n)^{-r+1} dx$$

Process to Form the Reduction Formulae:

A process to form reduction formula is given below for reduction formulae (1).

First Method: ***Using integration by parts:***

$\Psi(x) = \int x^m (a + bx^n)^r \, dx$

Multiplying and dividing by $nb\, x^{n-1}$ {because $(a + bx^n)' = nb\, x^{n-1}$

$$= \frac{1}{nb}\left[\int x^m \, . \, x^{-n+1}\,(nbx^{n-1})\,(a + bx^n)^r dx\right]$$

$$= \frac{1}{nb}\left[\int x^{m-n+1}\,(nbx^{n-1})(a + bx^n)^r dx\right]$$

$$= \frac{1}{nb}\left[\frac{1}{r + 1}\, x^{m-n+1}\,(a + bx^n)^{r+1} - \frac{m-n+1}{r+1}\int x^{m-n}\,(a + bx^n)^{r+1} dx\right]$$

$$= \frac{1}{nb}\left[\frac{1}{r + 1}\, x^{m-n+1}(a + bx^n)^{r+1} - \frac{m-n+1}{r+1}\int x^{m-n}\,(a + bx^n)^r(a + bx^n) dx\right]$$

$$= \frac{1}{nb\,(r + 1)}\, x^{m-n+1}\,(a + bx^n)^{r+1} - \frac{a(m-n+1)}{nb\,(r+)}\int x^{m-n}\,(a + bx^n)^r\, dx - \frac{m-n+1}{n\,(r+1)}\,\Psi(x)$$

$$\Psi(x) = \frac{1}{b\,(nr + m + 1)}\, x^{m-n+1}(a + bx^n)^{r+1} - \frac{a\,(m - n + 1)}{b\,(nr + m + 1)}\int x^{m-n}\,(a + bx^n)^r dx$$

where $P = \dfrac{1}{b\,(nr + m + 1)}$, $Q = \dfrac{-a\,(m - n + 1)}{b\,(nr + m + 1)}$

Second Method:

$$\Psi(x) = \int x^m \ (a + bx^n)^r \ dx$$

$g(x) = a + bx^n$ and $g'(x) = bnx^{n-1}$

To reduce the power of x.

$\lambda = 1 + \min\{m-n, n\} = m - n + 1$

$\mu = 1 + \min\{r, r\} = r + 1$

$$\int x^m \ (a + bx^n)^r dx = Px^{\lambda}(a + bx^n)^{\mu} + Q \int x^{m-n} \ (a + bx^n)^r dx$$

$$\int x^m \ (a + bx^n)^r dx = Px^{m-n+1}(a + bx^n)^{r+1} + Q \int x^{m-n} \ (a + bx^n)^r \ dx$$

Differencing with respect to x

$$x^m(a + bx^n)^r = P\,(m - n + 1)\, x^{m-n}(a + bx^n)^{r+1} + Pbn\,(r + 1)\, x^m(a + bx^n)^r + Q\, x^{m-n}(a + bx^n)^r$$

$$x^m(a + bx^n)^r = P\,(m - n + 1)\, x^{m-n}\,(a + bx^n)^r(a + bx^n) + Pbn\,(r + 1)\, x^m(a + bx^n)^r + Q\, x^{m-n}(a + bx^n)^r$$

$$x^m(a + bx^n)^r = [Pb\,(m - n + 1) + Pbn\,(r + 1)]\, x^m(a + bx^n)^r + [Pa\,(m - n + 1) + Q]\, x^{m-n}(a + bx^n)^r$$

By equating the coefficients, we get

$Pb\,(m - n + 1) + Pbn\,(r + 1) = 1 \quad , \quad Pa(m - n + 1) + Q = 0$

$$P = \frac{1}{b\,(nr + m + 1)} \quad \text{and} \quad Q = \frac{-a\,(m - n + 1)}{b\,(nr + m + 1)}$$

Example 5:

An object is moving so that its speed at time t is $v = t^5(t^2-3)^{-3/2}$ kilometer per hour.

Figure 10.16

(a) Find the distance of the particle at time t. At time 1.2 hours the particle is at distance 10 kilometer from the origin.

(b) Find the distance of the particle from the origin in 4 hours.

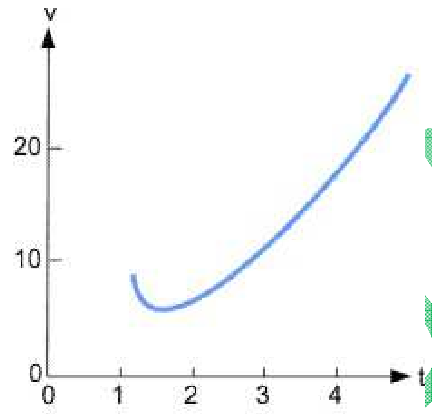

Figure 10.16

$Let \quad P = \frac{1}{m-2} \quad , \quad Q = \frac{m-1}{m-2}$

By (2):

If $m = 5$

$$\int t^5 \,(t^2-1)^{-\frac{3}{2}} dt$$

$$= \frac{1}{3} t^4 (t^2-1)^{-\frac{1}{2}} + \frac{4}{3}\int t^3 \,(t^2-1)^{-3/2} dt$$

If $m = 3$

$$\int t^3 \,(t^2-1)^{-\frac{3}{2}} dt$$

$$= t^2 (t^2-1)^{-\frac{1}{2}} + 2\int t\,(t^2-1)^{-\frac{3}{2}} dt$$

$$= t^2 (t^2-1)^{-1/2} - 2(t^2-1)^{-1/2}$$

Hence

$$\int t^5 \,(t^2-1)^{-\frac{3}{2}} dt$$

$$= \frac{1}{3} t^4 (t^2-1)^{-\frac{1}{2}} + \frac{4}{3} t^2 (t^2-3)^{-\frac{1}{2}} - \frac{8}{3}(t^2-3)^{-\frac{1}{2}} + C$$

$$x = \frac{1}{3\sqrt{t^2-1}}(t^4 + 4t^2 - 8) + C \qquad \rightarrow (3)$$

At $= 1.2$, $x = 10$, so by equation (3)

$$C = 10.08$$

Putting in equation (3)

$$x = \frac{1}{3\sqrt{t^2-1}}(t^4 + 4t^2 - 8) + 10.08 \qquad \rightarrow (4)$$

Figure 10.17

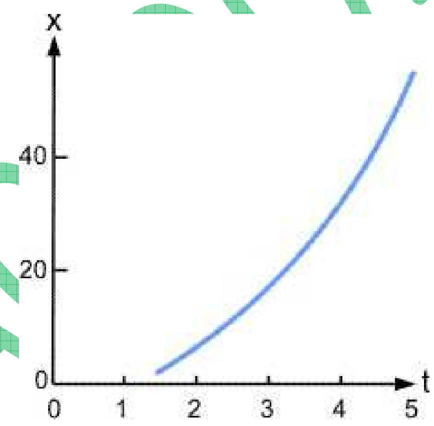

Figure 10.17

(b) Putting $t = 4$ in equation (4)

$$x = \frac{1}{3\sqrt{4^2-1}}(4^4 + 4.4^2 - 8) + 10.08$$

$$x = 36.93 \text{ km}$$