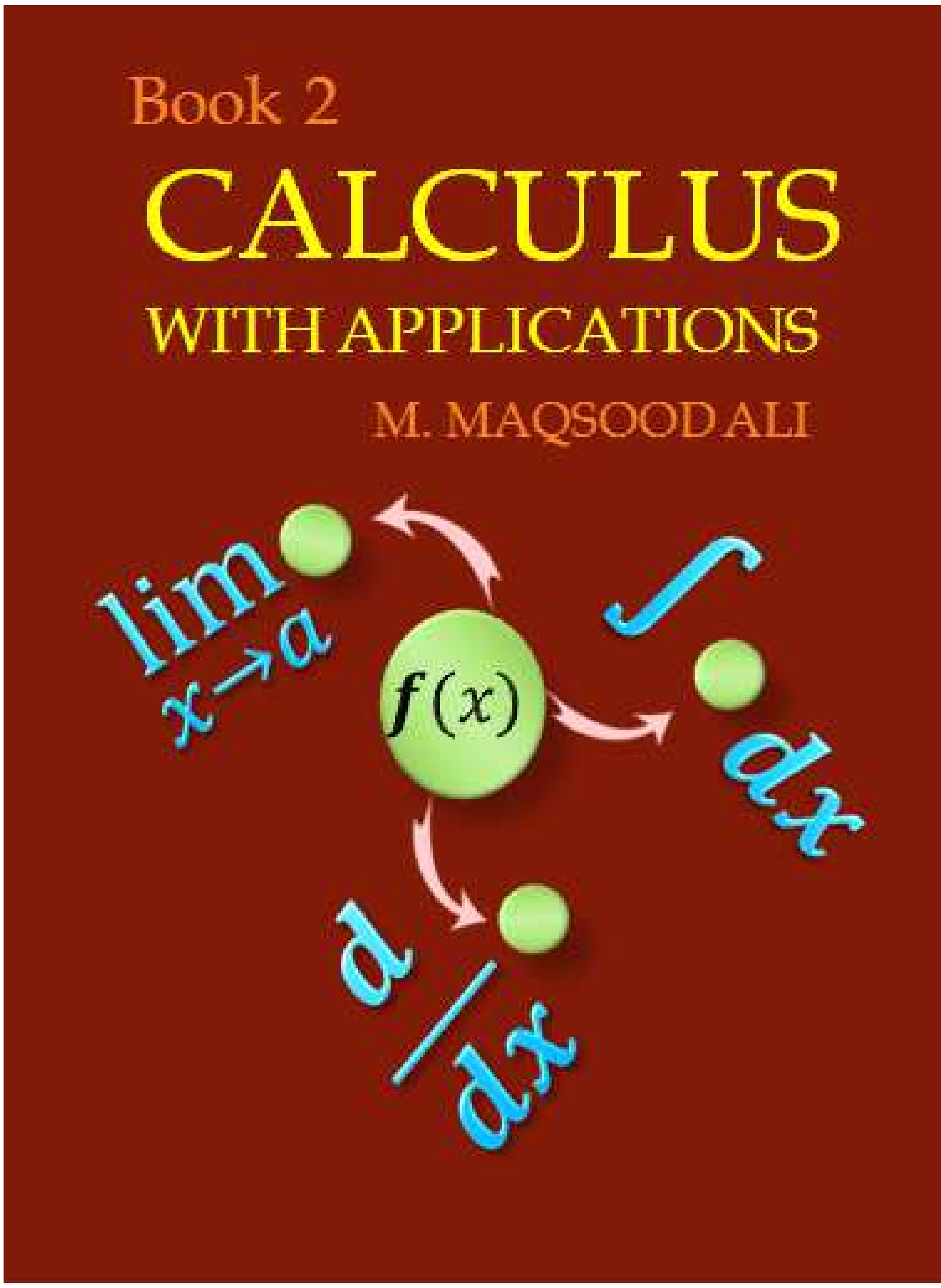
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
lim x→a
f(x)
∫ dx
d/dx

ANTIDERIVATIVES – INDEFINITE INTEGRALS:

If a function ϕ is defined every where the function f is itself then function ϕ is said to be an antiderivative of a function f if

$$\phi'(x) = f(x) \quad , \quad \text{for all} \quad x \in D_f$$

The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx$$

The symbol $\int$ is an integral operator and $f(x)$ is called integrand.

Theorem 1:

If $\phi(x)$ is an antiderivative of $f(x)$, then $\phi(x) + C$ is also an antiderivative of $f(x)$ for some constant c.

Proof:

Since $\phi(x)$ is an antiderivative of $f(x)$ therefore for all $x \in D_f$

$$\phi'(x) = f(x), \text{ then}$$

$$\frac{d}{dx}[\phi(x) + C] = \frac{d}{dx}\phi(x) + \frac{d}{dx}(C)$$

$$[\phi(x) + C]' = \phi'(x) + 0$$

$$[\phi(x) + C]' = f(x)$$

By the definition of antiderivative $[\phi(x) + c]$ is also the antiderivative of $f(x)$ for all $x \in D_f$.

Theorem 2:

If $\phi(x)$ and $\Psi(x)$ are two antiderivative of $f(x)$ for all $x \in D_f$ then the difference of $\phi(x)$ and $\Psi(x)$ must be a constant C.

$i.e.$ $\phi(x) - \Psi(x)$ = C

Proof:

As $\phi(x)$ and $\Psi(x)$ are two antiderivatives of a function $f(x)$ therefore

$\phi'(x) = f(x)$ and $\Psi'(x) = f(x) \qquad \forall\, x \in D_f$

Now

$$\frac{d}{dx}[\phi(x) - \Psi(x)] = \frac{d}{dx}\phi(x) - \frac{d}{dx}\Psi(x)$$

$$[\phi(x) - \Psi(x)]' = \phi'(x) - \Psi'(x)$$

$$[\phi(x) - \Psi(x)]' = f(x) - f(x)$$

$$[\phi(x) - \Psi(x)]' = 0 \quad , \quad \forall\, x \in D_f$$

which shows that $\phi(x) - \Psi(x)$ neither strictly increasing nor strictly decreasing therefore $\phi(x) - \Psi(x)$ is a constant C for all $x \in D_f$

$$\phi(x) - \Psi(x) = C$$

$$\phi(x) = \Psi(x) + C$$

This proves the theorem.

RULES FOR INDEFINITE INTEGRATION:

(1) The integral of the sum of $f(x)$ and $g(x)$ is equal to the sum of the integrals of $f(x)$ and $g(x)$.

Proof:

Suppose $\phi(x)$ and $\Psi(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively then

$$\phi'(x) = f(x) \quad \text{and} \quad \Psi'(x) = g(x)$$

We have

$$\frac{d}{dx}[\phi(x) + \Psi(x)] = \frac{d}{dx}\phi(x) + \frac{d}{dx}\Psi(x)$$

$$[\phi(x) + \Psi(x)]' = \phi'(x) + \Psi'(x)$$

$$[\phi(x) + \Psi(x)]' = f(x) + g(x)$$

Thus $\phi'(x) + \Psi'(x)$ is an antiderivative of $f(x) + g(x)$. Therefore we can write by using integral notation.

$$\int [f(x) + g(x)]dx = \phi(x) + \Psi(x)$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

(2) The integral of a constant C times a function $f(x)$ equals to the constant times the integral of the function.

Proof:

Suppose $\phi(x)$ is the antiderivative of a function $f(x)$ and C be any constant then $[C\phi(x)]' = Cf(x)$

$$\frac{d}{dx}[C\phi(x)] = C\frac{d}{dx}\phi(x)$$

$$[C\phi(x)]' = C\,\phi'(x)$$

$$[C\phi(x)]' = C\,f(x)$$

Thus $C\phi(x)$ is an antiderivative of $C\,f(x)$, Therefore by using integral notation we can write.

$$\int C\,f(x)dx = C\phi(x)$$

$$\int C\,f(x)dx = C\int f(x)dx$$

INTEGRATION BY SUBSTITUTION:

(1) **If the integrand is:**

(i) $f(g(x)g'(x)$, then substitution $u = g(x)$

(ii) $\frac{g'(x)}{f(g(x)}$, then substitution $u = g(x)$

(2) **Trigonometric substitution:**

If the integrand involves

(i) $(a^2 - x^2)^{n/2}$, then substitution $x = a\,sinu$

(ii) $(x^2 - a^2)^{n/2}$, then substitution $x = a\,secu$

(iii) $(x^2 + a^2)^{n/2}$, then substitution $x = a\,tanu$

(3) **Trigonometric substitution:**

If the integrand involves

(i) $\sqrt{a^2 + x^2}$, then substitution $x = a\,sinhu$

(ii) $\sqrt{x^2 - a^2}$, then substitution $x = a\,coshu$

(iii) $\sqrt{a^2 - x^2}$, then substitution $x = a\,tanhu$

Example 1:

A car is running on a straight road. The speed of the car is

$$v = \frac{-(t-8)^3}{10\sqrt{100-(t-8)^2}}$$

Figure 10.9

(a) Find the distance x (in kilometers) as a function of time t(in minutes), If the car start at a distance 60 km from the origin.

(b) Find the time when the car will stop.

(c) Find the distance covered by the car in 16 minutes

Solution:

Let x be the distance covered by the car in time t.

$$\frac{dx}{dt} = v$$

$$x = \int v\,dt$$

$$x = -\frac{1}{10}\int \frac{(t-8)^3}{\sqrt{100-(t-8)^2}}\,dt \quad \rightarrow (1)$$

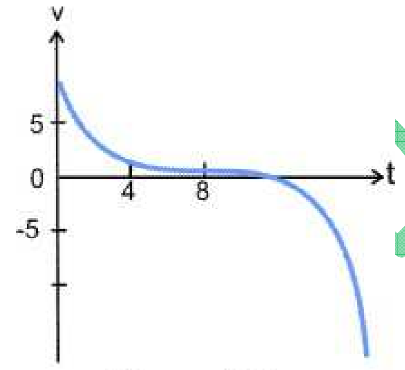

Figure 10.9

Let $t-8 = u \;\Rightarrow\; dt = du$

Putting in (1)

$$x = -\frac{1}{10}\int \frac{u^3}{\sqrt{10^2-u^2}}\,du \quad \rightarrow (2)$$

$Let\ u = 10\,sin\theta \;\Rightarrow\; du = 10\,cos\theta\, d\theta$

Putting in (2)

$$x = -100\int sin^3\theta\, d\theta$$

$$= -100\left\{-cos\theta - \frac{1}{3}cos^3\theta\right\} + C \quad \rightarrow (3)$$

since $u = 10\,sin\theta \;\Rightarrow\; sin\theta = \frac{u}{10}$, so

$$cos\theta = \sqrt{1-sin^2\theta} = \sqrt{1-\frac{u^2}{100}} = \frac{1}{10}\sqrt{100-u^2}$$

Putting in equation (2)

$$x = -\frac{1}{10}\left\{-100\sqrt{100-u^2} + \frac{1}{3}(100-u^2)^{3/2}\right\} + C$$

Putting $u = t-8$

$$x = 10\sqrt{100-(t-8)^2} - \frac{1}{30}(100-(t-8)^2)^{3/2} + C$$

$x = 60$ km, when $t = 0$ minutes, so

$$C = 7.2$$

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$$\int \frac{1}{A(t)} dA(t) = \int k\, dt$$

$$lnA(t) = kt + C \quad \rightarrow (1)$$

At $t = 0$, $A(t) = A(t_0)$

$$\ln A(t_0) = 0 + C$$

$$C = \ln A(t_0)$$

Putting in (1)

$$\ln A(t) = kt + \ln A(t_0)$$

$$\ln A(t) - \ln A(t_0) = kt$$

$$\ln \frac{A(t)}{A(t_0)} = kt$$

$$\frac{A(t)}{A(t_0)} = e^{kt}$$

$$A(t) = A_0\, e^{kt} \quad \rightarrow (2)$$

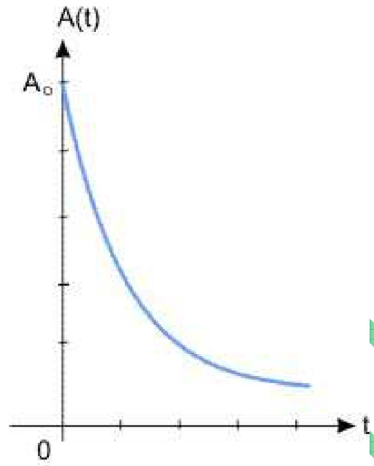

Figure 10.11

Figure 10.11

Firstly calculate the value of k using half life of carbon-14.

$$A(5730) = \frac{1}{2} A_0 \quad \rightarrow (3)$$

According (2)

$$A(5730) = A_0\, e^{5730k}$$

$$\frac{1}{2} A_0 = A_0\, e^{5730k}$$

$$0.5 = e^{5730k}$$

$$5730k = \ln 0.5$$

$$k = -0.000121$$

So that

$$A(t) = A_0\, e^{-0.000121t} \quad \rightarrow (4)$$

Since $A(t) = \frac{1}{10}A_0,$
so that by (4)

$$\frac{1}{10}A_0 = A_0 e^{-0.000121t}$$

$$\frac{1}{10} = e^{-0.000121t}$$

$$-\ln 10 = -0.000121t$$

$$t = 19030$$

The bone is about 19030 years old.

Example 3:

A bullet is fired vertically upward with a velocity 270 meters per second at a height 80 metres.
(a) Find the height of the bullet as a function of time.
(b) Find the height of the bullet from the ground at $t = 5$ seconds.

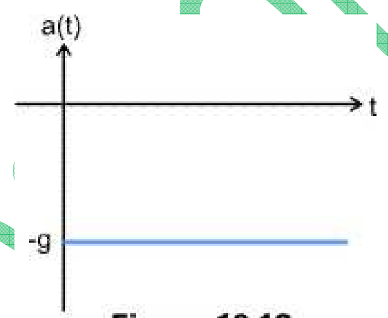

Figure 10.12

Solution:
(a) Acceleration due to gravity

$$a = -g \quad , \quad \textbf{Figure 10.12}$$

$$\frac{dv}{dt} = -g$$

$$v = -g\int dt$$

$$v = -gt + A \quad \rightarrow (1)$$

$v = 270\ m\backslash s$ at $t = 0$, putting in equation (1)

$$A = 270$$

Putting in equation (1)

$$v = -gt + 270 \quad \rightarrow (2)$$

Figure 10.13

is the velocity of the bullet at time t.

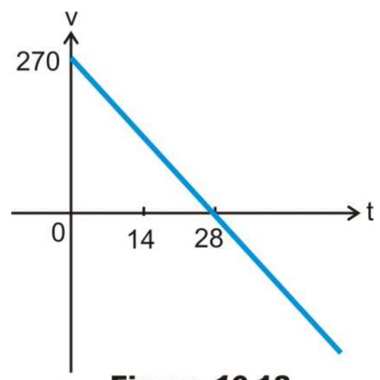

Figure 10.13

Let h be the height of the bullet at time t from the ground.

$$\frac{dh}{dt} = -gt + 270$$

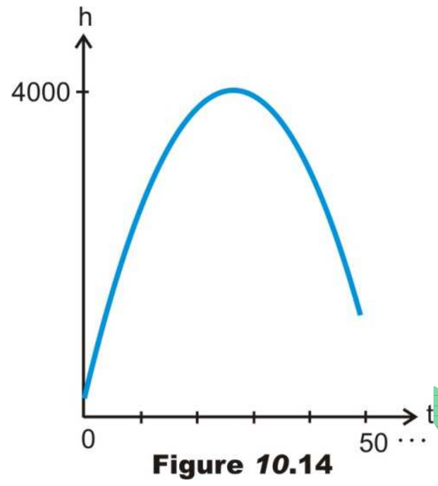

Figure 10.14

$$\int dh = \int (-gt + 270)dt$$

$$h = -\frac{1}{2}gt^2 + 270t + B \quad \rightarrow (3)$$

$h = 80$ metres $at\ t = 0$, putting in equation (3)

$$80 = 0 - 0 + B$$

$$B = 80$$

Putting in equation (3)

$$h = -\frac{1}{2}gt^2 + 270t + 80$$

is the height of the bullet at time t, **figure 10.14.**

(b) Putting $t = 5\ sec$

$$h = 1307.5 \text{ m}$$

The height of the bullet from the ground is 1307.5 metres at $t = 5$ seconds.

Example 4:

The marginal cost and marginal revenue functions for a company's product are

$$\overline{MC} = 0.06q + 2$$

$$\overline{MR} = -0.08q + 20$$

The fixed cost is $200 and the revenue is zero when no units are sold.

(a) Determine the total cost function for the product.

(b) Determine the total revenue function for the product.

(c) Determine the total profit function for the product.

Solution:

(a) $$C'(q) = \overline{MC} = 0.06q + 2$$

$$C(q) = \int (0.06q + 2)dq$$

$$= 0.03\,q^2 + 2q + A \quad \rightarrow (1)$$

$C = \$200$ for $q = 0$, because fixed cost is \$200, so by equation (1)

$$A = 200$$

Putting in equation (1)

$$C(q) = 0.03\,q^2 + 2q + 200$$

is the total cost function, **figure 10.15a**.

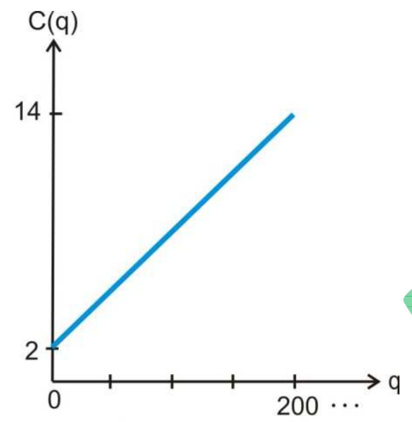

Figure *10*.15 a

(b) $$R'(q) = \overline{MR} = -0.08q + 20$$

$$R(q) = \int (-0.08q + 20)dq$$

$$= -0.04q^2 + 20q + B \quad \rightarrow (2)$$

$R = 0$ for $q = 0$, so by equation (2)

$$B = 0$$

Putting in equation (2)

$$R(q) = -0.04q^2 + 20q$$

is the total revenue function, **figure 10.15b**.

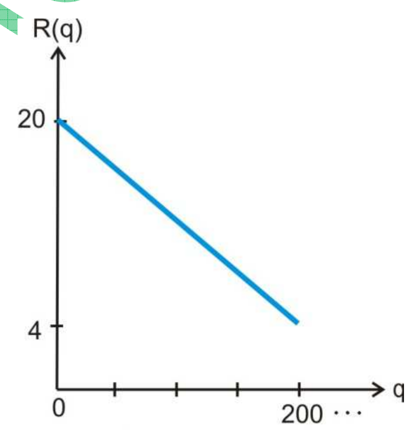

Figure *10*.15 b

(c) The total profit function $P(t)$ is

$$P(t) = R(t) - C(t)$$

$$= (-0.04q^2 + 20q) - (0.03\,q^2 + 2q + 200)$$

$$= -0.07q^2 + 18q - 200$$