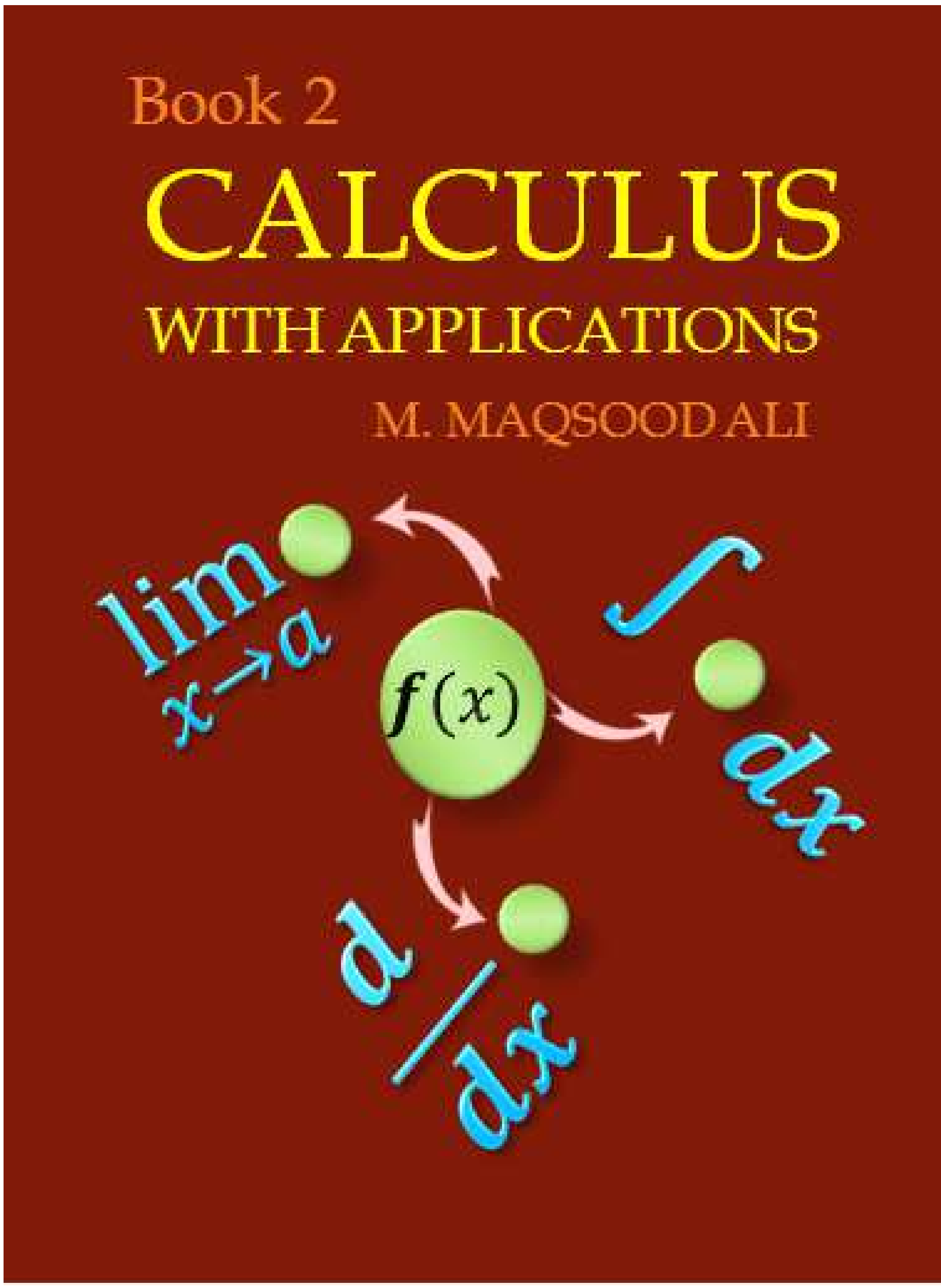
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x\to a}$
$f(x)$
$\int \, dx$
$\frac{d}{dx}$

EXAMPLES AND THEOREMS FOR TAYLOR AND MACLAURIN SERIES:

Theorem A-8:

If a series converges then its nth term has the limit zero as $n \to \infty$.

Proof:

Suppose that the series $\sum_{1}^{\infty} a_n$ is convergent.

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$

Hence,

$$S_n - S_{n-1} = a_n$$
$$\lim_{n\to\infty} S_n = l \text{ and } \lim_{n\to\infty} S_{n-1} = l$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$$
$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= l - l = 0$$

Theorem A-9:

$\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0$ for every real number x.

Proof:

Consider a series.

$$S_n = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

where x is a real number

Let $u_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty}\left[\frac{u_n + 1}{u_n}\right] = \lim_{n\to\infty}\left|\frac{x^{n+1}n!}{(n+1)!\,x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$$
$$= \lim_{n\to\infty}\left|\frac{1}{n+1}\right| \quad , \qquad |x| = 0 < 1$$

Since the limit 0 is less than 1 for every value of x, it follows from the ratio test that given series is absolutely convergent for all real numbers.

The preceding theorem state that if a series converges then its nth term has the limit 0 as $n \to \infty$,

$$\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = 0$$

Example 9.7:

Expend e^x in power of x.

Solution:

$$f(x) = e^x$$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1
5	e^x	1
$\vdots$	$\vdots$	
$n+1$	e^x	

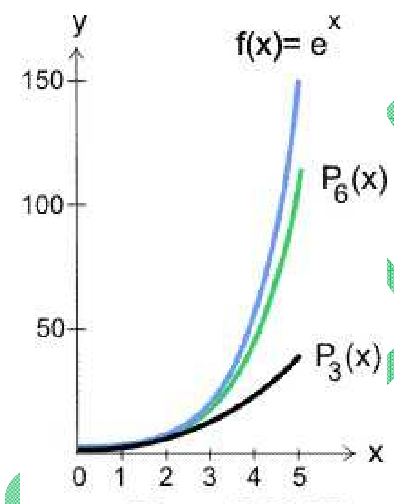

Figure 7.31 FF

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} e^c\right| = e^c \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\right|$$

Since $\lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\right| \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} |R_n| \to 0$$

Series of e^x in power of x converge to zero.
Maclaurin's series is valid for e^x.

$$f(x) = f(0+x)$$

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \frac{x^4}{4!} f^{(4)}(0) + \frac{x^5}{5!} f^{(5)}(0) + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}$$

Example 9.8:

Expand six x in power of x

Solution:

$$f(x) = \sin x$$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$sinx$	0
1	$cosx$	1
2	$-sinx$	0
3	$-cosx$	-1
4	$sinx$	0
5	$cosx$	1
$\vdots$	$\vdots$	
$n+1$	$\pm sinx$ or $\pm cosx$	

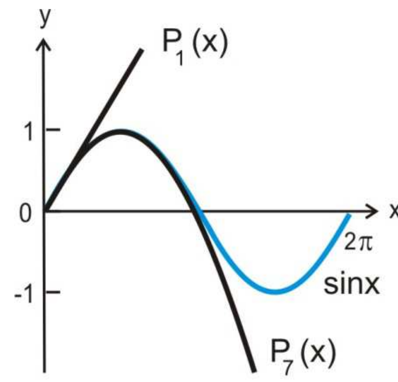

Figure 7.32 a

$$\left|f^{(n+1)}(c)\right| = |\pm sinc| = |sinc| \leq 1$$

and

$$\left|f^{(n+1)}(c)\right| = |\pm cosc| = |cosc| \leq 1$$

In both cases

$$\left|f^{(n+1)}(c)\right| \leq 1$$

Now

$$\lim_{n\to\infty}|R_n(x)| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c)\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\right| \lim_{n\to\infty}\left|f^{(n+1)}(c)\right|$$

$$\leq 0 \times 1$$

$$= 0$$

$R_n(x) \to 0 \; as \; n \to \infty$

Series converge to zero. Maclaurin's series is valid for $\sin x$.

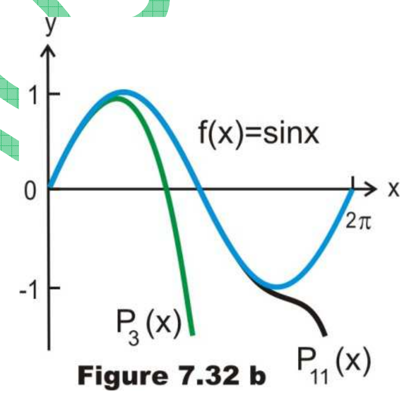

Figure 7.32 b

$f(x) = f(0 + x)$

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + \cdots$$

$$sin\, x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 9.9 :

Expand $f(x) = \sin x$ in power of $\left(x - \frac{\pi}{2}\right)$.

Solution:

$f(x) = \sin x$

k	$f^{(k)}(x)$	$f^{(k)}(\pi/2)$
0	$sinx$	1
1	$cosx$	0
2	$-sinx$	-1
3	$-cosx$	0
4	$sinx$	1
5	$cosx$	0
⋮	⋮	
$n+1$	$\pm sinx$ or $\pm cosx$	

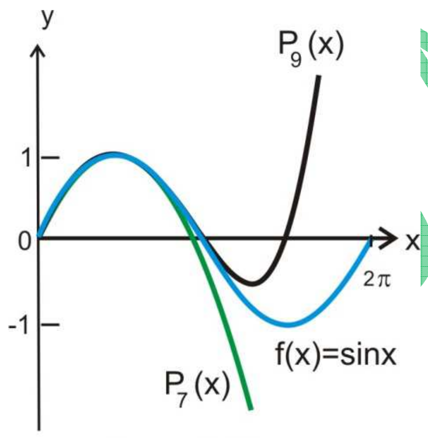

Figure 7.33

$$\left|f^{(n+1)}(c)\right| = |\pm sinc| = |sinc| \leq 1$$

and

$$\left|f^{(n+1)}(c)\right| = |\pm cosc| = |cosc| \leq 1$$

In both cases

$$\left|f^{(n+1)}(c)\right| \leq 1$$

Now

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{(x - \frac{\pi}{2})^{n+1}}{(n+1)!} f^{(n+1)}(c) \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x - \frac{\pi}{2})^{n+1}}{(n+1)!} \right| \lim_{n\to\infty} \left|f^{(n+1)}(c)\right|$$

$$\leq 0 \times 1$$

$$= 0$$

$R_n(x) \to 0 \ as \ n \to \infty$

The series of $\sin x$ in power of $\left(x - \frac{\pi}{2}\right)$ converge to zero.

Taylor's series is valid for $\sin x$.

The series of e^x in power of $(x-1)$ converge to zero.

Taylor's series is valid for e^x.

$$f(x) = f(1 + (x - 1)$$
$$= f(1) + (x-1)f^{(1)}(1) + \frac{(x-1)^2}{2!}f^{(2)}(1) + \frac{(x-1)^3}{3!}f^{(3)}(1) + \cdots$$

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2!}e + \frac{(x-1)^3}{3!}e + \cdots$$

$$e^x = e\{1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\}$$

Example 9.11:

Use an infinite series to approximate

$$\int_0^1 sinx^2 dx$$

to three decimal places

Solution:

By Maclaurin's series

$$sinx^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

So that

$$\int_0^1 \sin x^2 dx = \int_0^1 \{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\}dx$$

$$\int_0^1 sinx^2 dx = \left[\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^4}{1320} - \cdots\right] = 0.310$$

EXERCISE 1

(1) Write down the first n terms, remainder and the general form of order n of Maclaurin polynomial to the function $f(x) = cosx$.

(a) Using Maclaurin polynomial find and approximate value of $\cos\pi/4$, correct to five decimal places, for $n = 5$.

(b) (i) Find remainder and approximate value of $cos\pi/6$, for $n = 3$.

(ii) Find the value of c such that $cos\ \pi/6 = 0.5$.

(c) Prove that $f(x) = P_5(x)$, correct to one decimal places for all $x \in (0, \frac{\pi}{2})$.

(a) Using Maclaurin polynomial, find an approximate value of e^4, correct to three decimal places, for $n = 5$.

(b) Find the remainder and approximate value of e^7, correct to four decimal places, for $n = 6$ such that $e^7 = 1096.3316$.

(4) Find Macluarin polynomial and remainder in term of c of degree four for the function

$$f(x) = \frac{6}{x-4}$$

(a) Find the approximate value of $f(9)$ using Macluarin polynomial and also remainder such that $f(9) = 1.2$.

(c) Prove that $c \in (0,9)$.

(5) Find the general form of Taylor polynomial about 1 for the function

$$f(x) = lnx$$

(6) Find the general form of Taylor polynomial and remainder about a for the function

$$f(x) = lnx$$

(a) Find the approximate value and remainder (in term of c) of $ln10$ using Taylor's polynomial for $n = 5$ about 4 and 8.

(b) $ln10 = 2.302585$, correct to six decimal places, find the approximate value and remainder of $ln10$ by Taylor polynomial for n=5 about 9 and prove that $c \in (9,10)$.

EXERCISE 2

Use Taylor's series or Maclaurin's series to find the series of the following functions in power of indicated term.

(1) $f(x) = e^x$	in	x
(2) $f(x) = cosx$	in	x
(3) $f(x) = e^{x/a}$	in	$(x-a)$
(4) $f(x) = e^{-x}$	in	$(x+1)$
(5) $f(x) = ln(1-x)$	in	x
(6) $f(x) = ln(1+x)$	in	x
(7) $f(x) = lnx$	in	$(x-1)$

(8) Find a series in power of x for $f(x) = ln\,(k + x)$

(a) If $ln\ 10 = 2.3026$ then find

(i) $ln19$ (ii) $ln19$ (iii) $ln8$ (iv) can you find2

(b) If $ln\ 100 = 4.6052$ then find

(i) $ln199$ (ii) $ln10$ (iii) $ln150$ (iv) $ln90$

(v) can you find250

(9) Find a series for $ln(x/3)$ in power of $x-3$ and show that it is convergent by ratio method and valid for $0 < x < 6$.

(10) Find a series in power of x for the following and using ratio test to show that the series is convergent.

(i) $f(x) = sinhx$ (ii) $f(x) = coshx$

Use an infinite series to approximate the following to four decimal places:

(11) $\int_0^1 \frac{sinx^2}{x^2}dx$ (12) $\int_0^1 \frac{1-cosx^3}{x^2}dx$

(13) $\int_0^{0.5} x^2cosx^2dx$ (14) $\int_0^{0.5} \frac{\ln(1+x)}{x}dx$

(15) $\int_0^{0.1} \frac{(1-e^{x^2})}{x}dx$

Using infinite series approximately the following to four decimal places.

(16) $ln1.6$ (17) $cos1^o$ (18) $e^{1/2}$ (19) $\sqrt[3]{e}$