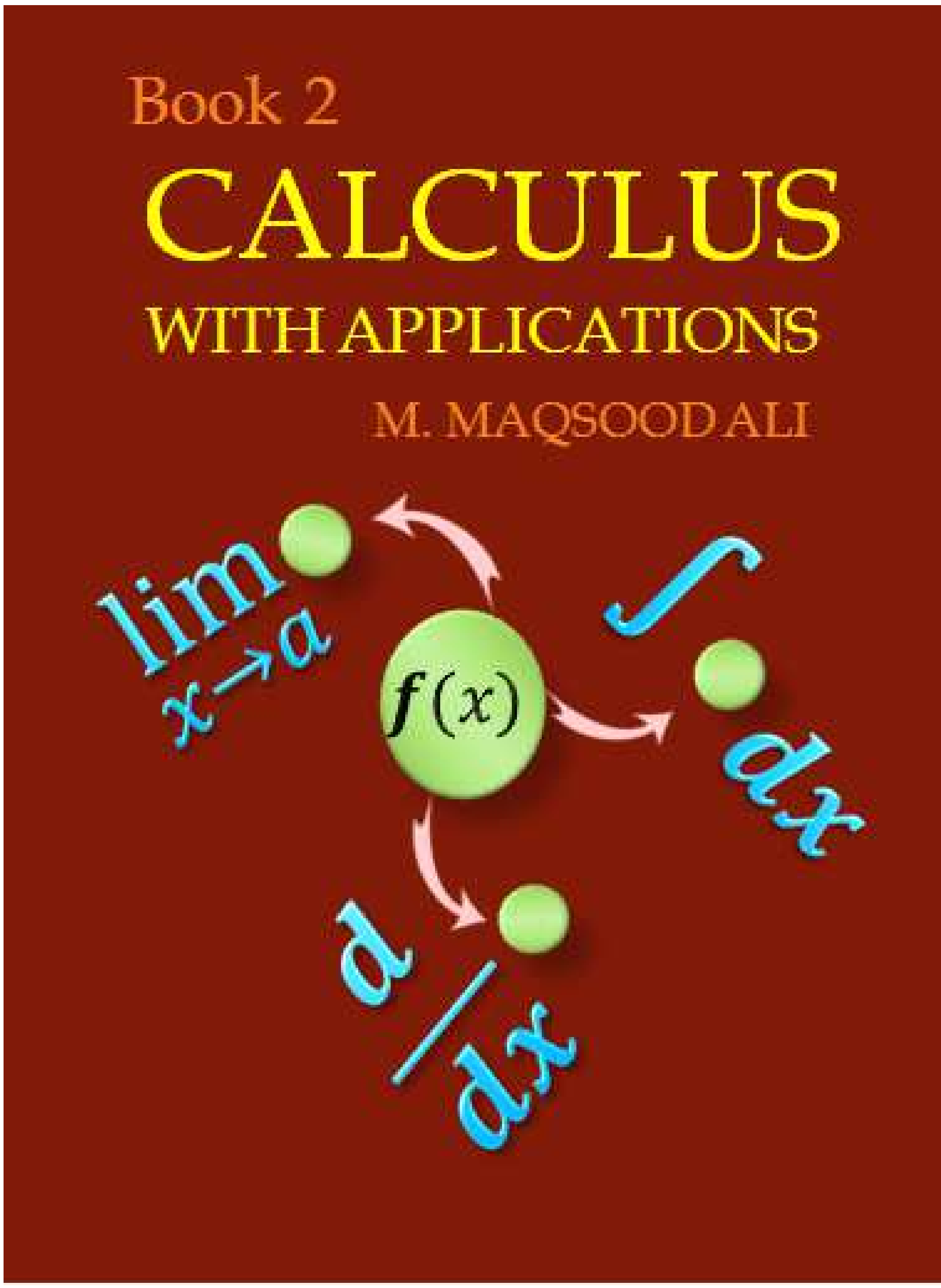
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
lim
x→a
f(x)
∫
dx
d
dx

Chapter 8

ROLL'S AND MEAN VALUE THEOREMS

STRICTLY INCREASING FUNCTIONS:

A function f is said to be strictly increasing on a set S subset ℝ if

$f(x_1) < f(x_2)$ for all $x_1, x_2 \in S$ such that $x_1 < x_2$

The value of a strictly increasing function $f(x)$ continue increases on increasing the value of x.

The graph of an strictly increasing function $f(x)$ rises up as x moves to the right, as shown in the

figure 8.1.

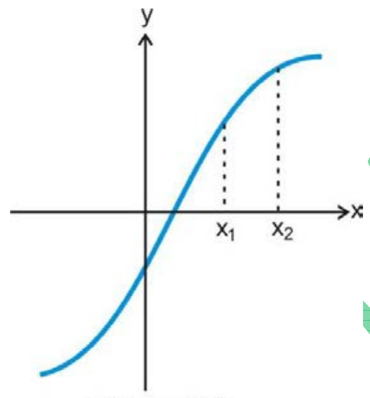

Figure 8.1

Examples 8.1:

(i) $f(x) = 5x$ is a strictly increasing function on the set of real numbers, because

$f(1) < f(2)$ for $1 < 2$

$f(2) < f(3)$ for $2 < 3$

$f(x_1) < f(x_2)$ for all $x_1 < x_2$

figure 8.2.

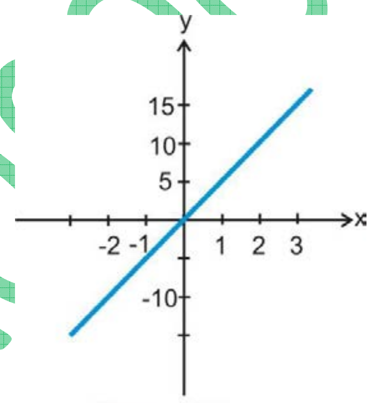

Figure 8.2

STEADILY INCREASING FUNCTIONS:

A function f is said to be steadily increasing on a set S subset of ℝ if

$f(x_1) \le f(x_2)$ for all $x_1, x_2 \in S$ such that $x_1 < x_2$

The value of a steadily increasing function $f(x)$ increase or remain unchanged on increasing the value of x as shown in

figure 8.3.

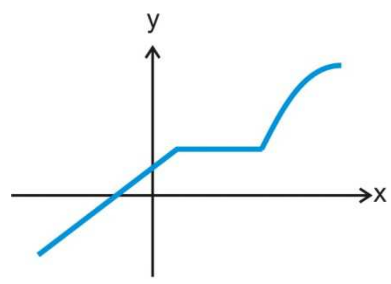

Figure 8.3

Examples 8.2:

$f(x) = \lfloor x \rfloor$ is steadily increasing function on the set of real numbers.

$$f(x) = \begin{cases} -1 & \text{for} \quad -1 \le x < 0 \\ 0 & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad 1 \le x < 2 \\ 2 & \text{for} \quad 2 \le x < 3 \\ 3 & \text{for} \quad 3 \le x < 4 \end{cases}$$

Since $f(0) < f(1)$ for $0 < 1$

$f(0) = f(0.5)$ for $0 < 0.5$

$\therefore$ f is steadily increasing.

Figure 8.4

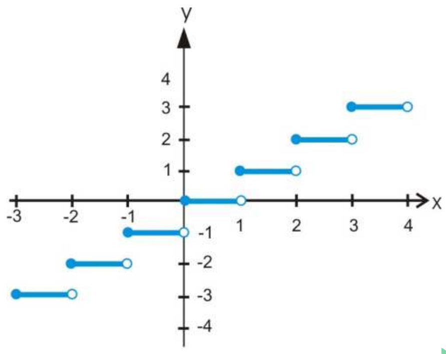

Figure 8.4

STRICTLY DECREASING FUNCTIONS:

A function f is said to be strictly decraeasing on a set S subset of ℝ if

$f(x_1) > f(x_2)$ for all $x_1, x_2 \in S$ such that $x_1 < x_2$.

The value of strictly decreasing function $f(x)$ continue decreases on increasing the value of x.

The graph of a strictly decreasing function falls as x moves right, as shown in

figure 8.5.

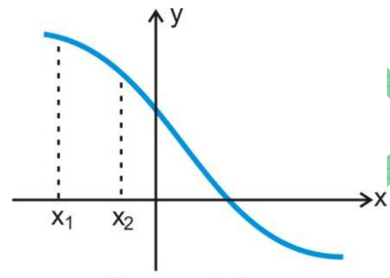

Figure 8.5

Examples 8.3:

$$f(x) = \frac{1}{x}$$

is a strictly decreasing function on the set of positive integers.

Since $f(1) > f(2)$ for $1 < 2$

$f(2) > f(3)$ for $2 < 3$

$f(x_1) > f(x_2)$ for all $x_1 < x_2$

figure 8.6.

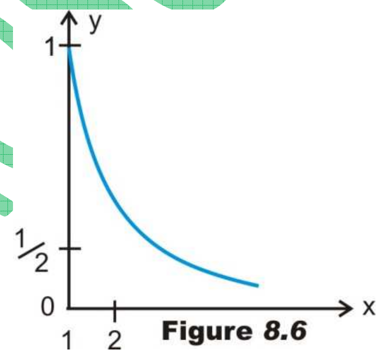

Figure 8.6

STEADILY DECRAEASING FUNCTIONS:

A function f is said to be steadily decreasing on a set S subset of ℝ if

$f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in S$ such that $x_1 < x_2$

The value of a steadily decreasing function $f(x)$ decrease or remain unchanged on increasing the value of x as shown in

figure 8.7.

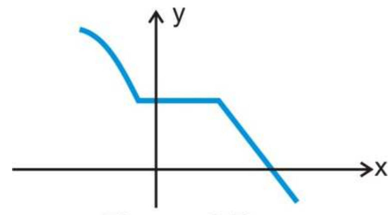

Figure 8.7

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Thus f is strictly increases in the neighbourhood of a,
Conversely, f is strictly increasing in an open interval $(a, a + \delta)$.

$\Rightarrow \; f(x) > f(a)$ for $x > a$, $\quad x \epsilon (a, a + \delta)$

$\Rightarrow \; f(x) - f(a) > 0$ for $x - a > 0$

$$\Rightarrow \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} > 0$$

$$\Rightarrow \quad f'(\text{a}) > 0 \qquad \to \text{(i)}$$

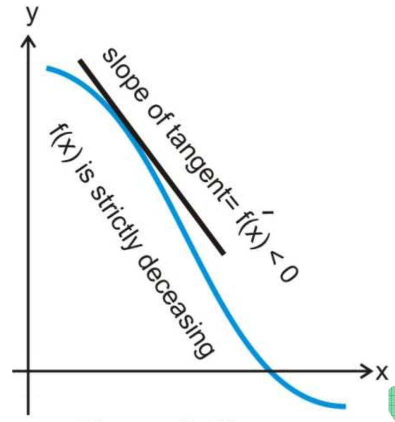

Figure 8.10

f is strictly increasing on an open interval $(a - \delta, a)$

$\Rightarrow \quad f(x) < f(a)$ for $x < a$; $x \in (a - \delta, a)$

$\Rightarrow \quad f(x) - f(a) < 0$ for $x - a < 0$

$$\Rightarrow \quad \frac{f(x) - f(a)}{x - a} > 0$$

$$\Rightarrow \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} > 0$$

$$\Rightarrow \quad f'(\text{a}) > 0 \qquad \to \text{(ii)}$$

Combining above two results we can say that $f'(a) > 0$ if and only if f is strictly increasing in some neighbourhood of a $\; i.e \; (a - \delta, a + \delta)$.

Figure 8.9.

(ii) The proof is left for the reader. **Figure 8.10**.

Example 8.4:

Show that the value of f is strictly increasing for all $x \in (-\infty, -2)$ and for all $x \in (3, \infty)$ but strictly decreasing for all $x \in (-2\,, 3)$.

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 5$$

Solution:

$$f'(x) = x^2 - x - 6 = (x - 3)(x + 2)$$

f is strictly increasing when

$$f'(x) > 0$$

$\Rightarrow \quad (x-3)(x+2) > 0$

It is possible when both factors have same signs.

Case-I:

For strictly increasing:

Suppose both factors are non-negative.

$x - 3 > 0$ and $x + 2 > 0$

$x > 3$ and $x > -2$

$x \in (3, \infty)$ and $x \in (-2, \infty)$

$x \in (-2, \infty) \cap (3, \infty) \Rightarrow x \in (3, \infty)$

Case II: Suppose both factors are negative.

$x - 3 < 0$ and $x + 2 < 0$

$x < 3$ and $x < -2$

$x \in (-\infty, 3)$ and $x \in (-\infty, -2)$

$x \in (-\infty, 3) \cap (-\infty, -2)$

$x \in (-\infty, -2)$

Hence f is strictly increasing for all $x \in (-\infty, -2)$ and for all $x \in (3, \infty)$.

For strictly decreasing:

f is strictly decreasing when

$$f'(x) < 0.$$

$$(x-3)(x+2) < 0.$$

It is possible when both the factors have opposite signs.

Case-I:

Suppose $(x-3)$ is negative and $(x+2)$ is positive

$x - 3 <$ and $x + 2 > 0$

$x < 3$ and $x > -2$

$x \epsilon (-\infty, 3)$ and $x \epsilon (-2, \infty)$

$x \epsilon (-\infty, 3) \cap (-2, \infty)$

$\Rightarrow \quad x \epsilon (-2,3)$

Case-II:

Suppose $(x-3)$ is positive and $(x+2)$ is negative.

$x - 3 > 0$ and $x + 2 < 0$

$x > 3$ and $x < -2$

$x \epsilon (3, \infty)$ and $x \epsilon (-\infty, -2)$

$x \epsilon (3, x) \cap (-x, -2)$

$\Rightarrow \quad x \epsilon \{ \}$ Hence according to case 1 and 2 , f is strictly decreasing for $x \epsilon (-2,3)$.

ROLLE'S THEOREM

Statement: f is a function with domain $[a, b]$, if

(i) f is continuous on $[a, b]$

(ii) f is derivable in (a, b)

(iii) $f(a) = f(b)$

then there exists at least one real number

c ϵ (a, b) such that $f'(c) = 0$.

Proof:

f is continuous on $[a, b]$

f is bounded and attains its bounds {by theorem A-2}

Let M and m be the least upper bound and greatest lower bound of $f(x)$.

We have two values c,d ϵ $[a, b]$ such that $f(c) = M$ and $f(d) = m$

Thus we have two case

Case-I:

$f(x)$ is equal to $f(a)$ and $f(b)$.

$$f(x) = f(a) = f(b) \;,\; x\,\epsilon\,[a, b]$$

Figure 8.11

In this case f is a constant function in the interval $[a, b]$.The derivative of a constant function must be equal to zero.

$$f'(x) = 0 \text{ for all } x\,\epsilon\,[a, b]$$

The theorem is true in this case.

Case-II:

$f(x)$ is a variable function.

$\Rightarrow$ $m \neq M$ but $f(a) = f(b)$

$\Rightarrow$ At least one of M and m must be different from $f(a)$ and $f(b)$, then there are following three cases arises.

(i) $m = f(a) = f(b)$ but $M \neq f(a) = f(b)$

Figure 8.12a

(ii) $m \neq f(a) = f(b)$ but $M = f(a) = f(b)$

Figure 8.12b

(iii) $m \neq f(a) = f(b)$ and $M \neq f(a) = f(b)$

Figure 8.12c

Consider the case M $\neq f(a) = f(b)$

Since $f(c) = M \geq f(x)$ for all $x\,\epsilon\,[a, b]$

$\Rightarrow$ At $x = c$ the point is a critical point.

$\Rightarrow$ f is neither strictly increasing nor strictly decreasing at $x = c$.

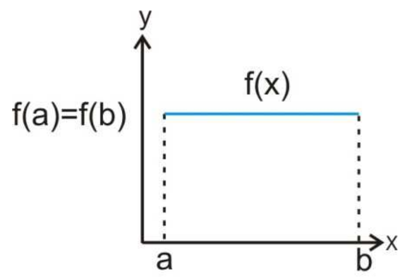

Figure 8.11

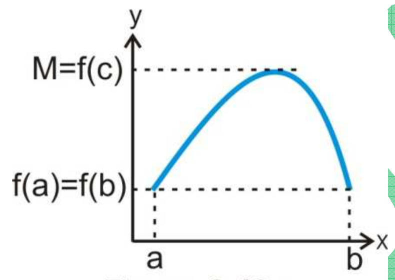

Figure 8.12 a

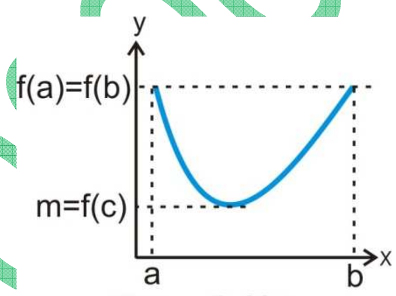

Figure 8.12 b

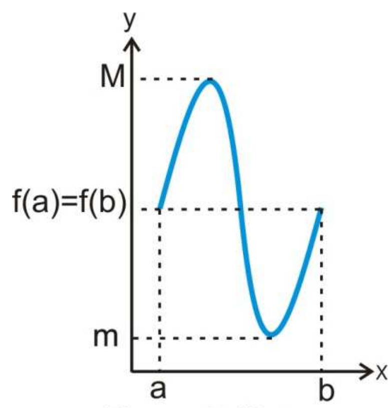

Figure 8.12 c

If we suppose f is strictly increasing at $x = c$, then we can prove it wrong by taking two points c and $c + \varepsilon$ in $[a, b]$.

Since M is least upper bound of f.

$\Rightarrow$ $M = f(c) > f(c + \varepsilon)$ for $c < c + \varepsilon$

$\Rightarrow$ f is not strictly increasing at $x = c$

$\Rightarrow$ If we suppose f is strictly decreasing at $x = c$, then we can also prove it is wrong by taking two points $c - \varepsilon$ and c in [a,b]

Since M is least upper bound of f.

$M = f(c) > f(c - \varepsilon)$ for $c > c - \varepsilon$

$\Rightarrow$ f is not strictly decreasing at $x = c$,

Hence f is neither strictly increasing nor srictly decreasing at $x = c$.

$\Rightarrow$ $f'(c) \ngtr 0$ and $f'(c)$ 0{by theorem $A - 7$}

$\Rightarrow$ $f'(c) = 0$ for $c \in (a, b)$

Similarly we can prove for $m \neq f(a) = f(b)$

GEOMETRICAL INTERPRETATION:

If a function f is continuous on $[a, b]$ and derivable on (a, b) and $f(a) = f(b)$, then there exists at least one real number c $\in (a, b)$ where the tangent is parallel to x-axis or slope of the tangent is equal to zero.

(i) **Figure 8.13a:** At least one real number c $\in (a, b)$ where the tangent is parallel to x-axis or slope of the tangent is equal to zero.

(ii) **Figure 8.13b:** More than one real numbers c, d $\in (a, b)$ where the tangents are parallel to x-axis or slope of the tangents is equal to zero.

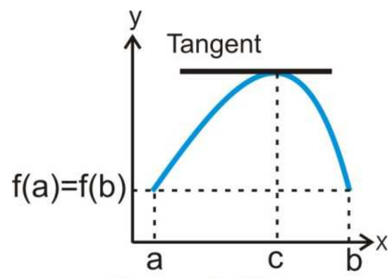

Figure 8.13 a

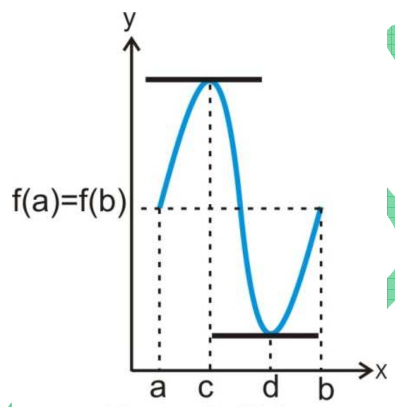

Figure 8.13 b

Algebraically:

If a and b are two real roots of $f(x) = 0$, then there exists at least one real root of the polynomial $f'(x) = 0$ lies between a and b

Example 8.5:

Verify Rolle's theorem for the following function

$f(x) = \sin x \;\; ; \;\; x \in (0, \pi)$

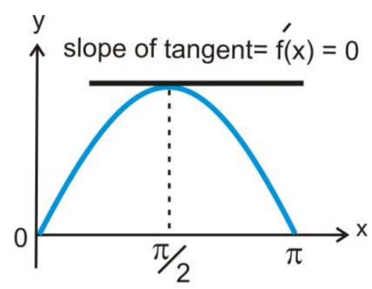

Figure 8.14

Solution:

$$f(x) = \sin x \quad ; \quad x \in [0, \pi]$$

$$f(0) = 0 \; and \; f(\pi) = 0$$

$f(x)$ is continuous on $[o, \pi]$ and derivable in $(0, \pi)$ and

$$f(0) = f(\pi).$$

So that the conditions of Rolle's theorem are satisfied.

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LAGRANGE'S MEAN VALUE THEOREM

Statement: Let f is a function with domain $[a, b]$,if

(i) f is continuous on $[a, b]$ and

(ii) f is derivable on (a, b)

Then there exists at least one real number c ϵ (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof:

Let ϕ is a function defined as

$$\phi(x) = A\,f(x) + Bx \quad \to (1)$$

where A and B are constants we choose A and B such that

$$\phi(a) = \phi(b)$$

From(1) $A\,f(a) + Ba = A\,f(b) + Bb$

$$A\,f(a) - A\,f(b) = Bb - Ba$$

$$-A\,\{f(b) - f(a)\} = B(b - a)$$

$$\frac{f(b) - f(a)}{b - a} = -\frac{B}{A} \quad \to (2)$$

Since $f(x)$ is continuous on [a,b]and derivable on (a,b) so $\phi(x)$ is continuous on [a,b] and derivable on(a,b) and $\phi(a) = \phi(b)$, then by Rolle's theorem there exists at least one real number c ϵ (a, b) such that

From(1) $\phi'(x) = A\,f'(x) + B$

$$\phi'(c) = A\,f'(c) + B$$

$$0 = A\,f'(c) + B$$

$$f'(c) = -\frac{B}{A} \quad \to (3)$$

On equating (2) and (3),we get

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

which prove the theorem.

ANOTHER FORM OF THE THEOREM:

Suppose that $b = a + h, h > 0 \Rightarrow [a, b] = [a, a + h]$

Let θ be positive number less than 1 so $0 < \theta < 1$.

$\Rightarrow \quad \theta h < h \quad \Rightarrow \quad a + \theta h < a + h$

$\Rightarrow \quad a + \theta h\ \epsilon\ (a, a + h)$

Let $c = a + \theta h\ \epsilon\ (a, a + h)$

Substituting these values in equation (4) we get

$$f'(a + \theta h) = \frac{f(a + h) - f(a)}{h}\ ,\ \ 0 < \theta < 1$$

GEOMETRICAL INTERPRETATION:

If f is continuous on $[a, b]$ and derivable on (a, b), then there exists at least one real number c ϵ (a, b), where the tangent is parallel to the chord which join the points $(a, f(a))$ and $(b, f(b))$.

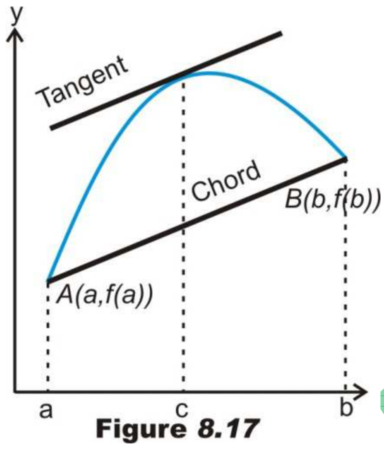

Figure 8.17

Figure 8.17

Slope of the chord joining the points $(a, (f(a))$ and $(b, f(b))$ is equal to the slope of tangent at $c \in (a, b)$.

$$Slope\ of\ the\ chord = \frac{f(b) - f(a)}{b - a}$$

$$Slope\ of\ the\ tangent = f'(c)$$

Example 8.8:

Apply mean value theorem to find $c \in (-1,1)$ when

$$f(x) = 5x^3 - 3x^2 + 9x + 2, \quad a = -1\,, b = 1$$

Solution:

$$f(x) = 5x^3 - 3x^2 + 9x + 2 \quad \rightarrow (1)$$

$$f'(x) = 15x^2 - 6x + 9$$

$$f'(c) = 15c^2 - 6c + 9$$

$f(a) = f(-1) = -15$

$f(b) = f(1) = 13$

By mean value of theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$15c^2 - 6c + 9 = \frac{13 + 15}{2}$$

$$15c^2 - 6c - 5 = 0$$

$$c = \frac{3 \pm 2\sqrt{21}}{15} \in (-1,1)$$

CAUCHY'S MEAN VALUE THEOREM

Statement:

If two functions f and g with domain $[a,b]$ are continuous on $[a,b]$ and derivable on (a,b) and if $f'(x)$ and $g'(x)$ do not both vanish for the same value of $x \in (a,b)$ then there exists at least one number $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof:

Suppose that Ψ is a fuction defined as

$$\Psi(x) = A\,f(x) + B\,g(x) \qquad \rightarrow (1)$$

where A and B are constants.

Substituting $x = a$ and $x = b$ in (1) we get

$$\Psi(a) = A\,f(a) + B\,g(a)$$
$$\Psi(b) = A\,f(b) + B\,g(b)$$

Choose A and B such that

$$\Psi(a) = \Psi(b)$$
$$A\,f(a) + B\,g(a) = A\,f(b) + B\,g(b)$$
$$A\,\{f(b) - f(a)\} = -B\,\{g(b) - g(a)\}$$
$$\frac{f(b)-f(a)}{g(b)-g(a)} = -\frac{B}{A} \qquad \rightarrow (2)$$

Since f and g are continuous on $[a,b]$ and derivable on (a,b) so also $\Psi\,(x)$.

$$\Psi'(x) = A\,f'(x) + B\,g'(x)$$

Ψ is continuous on $[a,b]$, derivable on (a,b) and $\Psi(a) = \Psi$(b), by Rolle's theorem there is at least one number $c \in (a,b)$ such that

$$\Psi'(c) = 0$$
$$Af'(c) + B\,g'(C) = 0$$
$$\frac{f'(c)}{g'(c)} = -\frac{B}{A} \qquad \rightarrow (3)$$

According to (2) and (3)

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(C)}{g'(c)}\,, \qquad c \in (a\,,b)$$

EXERCISE

(1) Show that the value of f is strictly increasing for all $x \in (-\infty,-3)$ and for all $x \in (2,\infty)$ but strictly decreasing for all $x \in (-3,-2)$.

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 3$$

(2) Show that

$$x < sinx < -\frac{1}{6}x^3$$

Verify Rolle's theorem for the following

(3) $f(x) = \frac{cosx}{e^x} \quad in \ (-\frac{\pi}{2}, \frac{\pi}{2})$

(4) $f(x) = log\frac{x^2+4}{5x} \ in\ (1\,,4\,)$

(5) $f(x) = sinx + cosx \ in\ (0, \pi/2)$

(6) $f(x) = e^{x^2} \ in\ (-1\,,1)$

Show that Rolle's theorem is not valid for the following

(7) $f(x) = (x - a)^{2/3} \ in\ (\,0\,, 2a)$

(8) $f(x) = x^{4/5} \ in\ (\,-1, 1)$

(9) $f(x) = log[secx + tanx] \ in\ (0\,, \pi)$

(10) $f(x) = sinx \ in\ (0\,, \pi/2)$

Show that following equations has at least one root in the given interval.

(11) $2cos2x + sin\,x = 0\,; \ (\pi/2, \pi/2)$

(12) $9x^8 + 10x^4 - 3x^2 + 8x - 6 = 0; (0,1)$

(13) If $f(x) = (5 - x^2)\,log\,x$ then show that the equation $log\,x = (5 - x)/2x$ is satisfied by at least one value of x lying between 1 and 5.

(14) Find c in the law of mean if $f(x) = tan^{-1}x\,, a = 1\ and\ b = \sqrt{3}$ and show that it lies in the required interval.

(15) Find c in the mean value theorem for the function e^x in the interval (2,3).

(16) Whether the mean value theorem is applicable or not for the function $f(x) = (x^2 + 1)x$ in the interval $(-1,1)$.

(17) Find the value of c in the main value if

$$f(x) = x^4 - 5x\ ; a = 0\,, b = \frac{1}{2}$$

(18) If $f(x) = x^3 + 1$ and $g(x) = x^2$ in the interval [0,1] determine the constant c in Cauchy's mean value theorem.

(19) If $f(x) = sinx$ and g(x) = cosx in the interval $[-\pi/4, \pi/4]$ determine the constant c in Cauchy's mean value theorem.