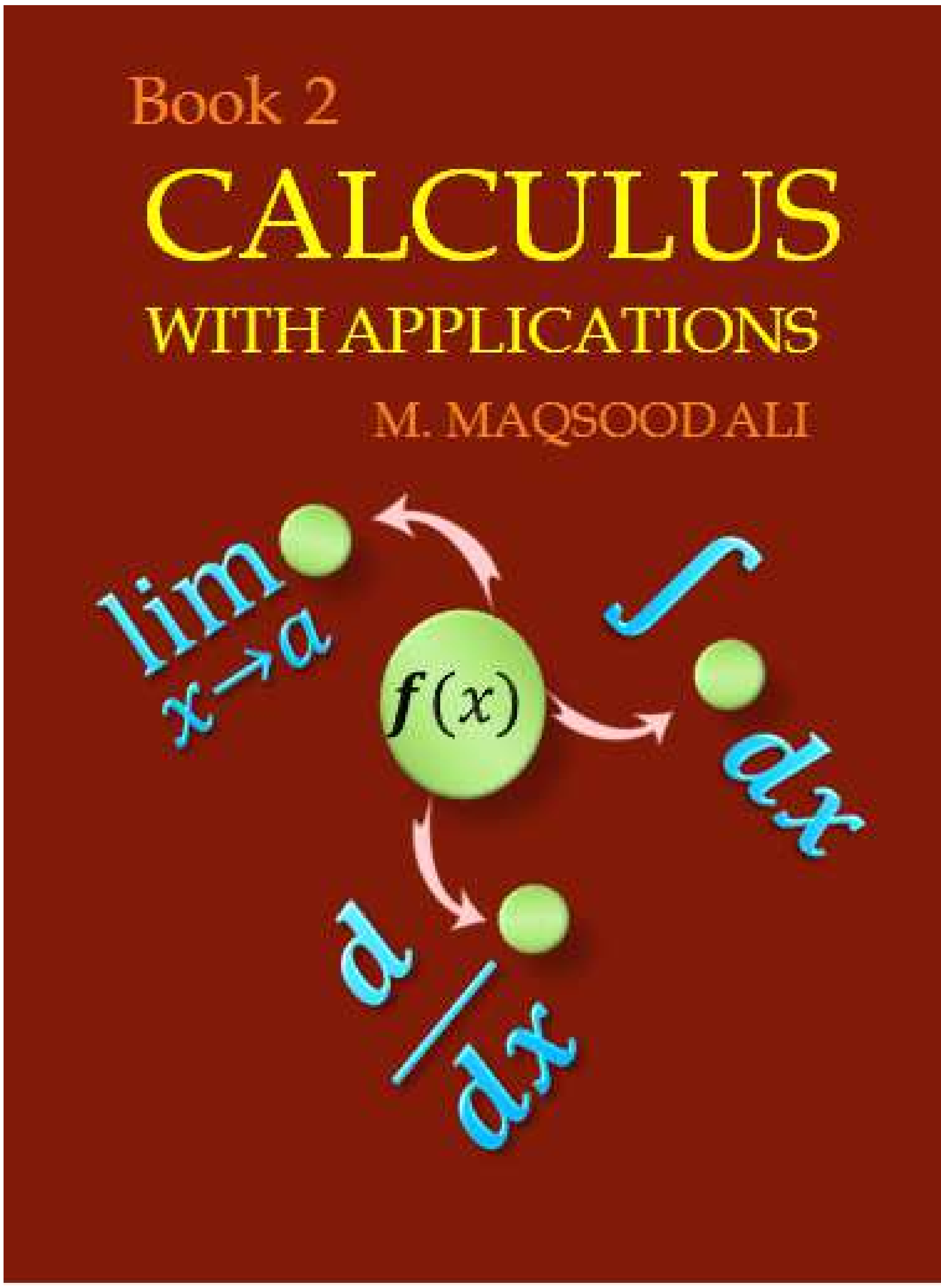
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

MAXIMA AND MINIMA

CRITICAL POINTS

The study of the graph of a function tells us about the behavior of the function that where is function has maximum point and minimum point. It also tells us about the shape of the curve that in a internal the curve is concave up or concave down or a straight line.

Without drawing the graph it is difficult to tell about the behavior of the function. But derivatives solve our problems without drawing the curves. Before study of these topics it is very important to know about some particular points lie on a curve.

Critical, Stationary, Inflection Points and Cusp:

"Critical points are the points where $f'(x) = 0$ or $f(x)$ is not differentiable $\left(\frac{1}{f'(x)} = 0\right)$."

Critical points consist of two types of points.

(1) Points where $f'(x) = 0$.

These points are called stationary points.

The stationary points consist of two types of points.

(i) Turning points (rel. min points and rel. max points).

(ii) Those points of inflection where $f'(x) = 0$.

(2) Points where $f(x)$ is not differentiable or $\frac{1}{f'(x)} = 0$.

These points are cusp or those points of inflection where $\frac{1}{f'(x)} = 0$.

Points of Inflection:

(1) A point $(a, f(a))$ is called point of inflection if

$$f''(a) = 0 \quad \text{or} \quad \frac{1}{f''(x)} = 0$$

but

$$f^{(n)}(a) \neq 0 \quad ; \quad n = 3, 5, 7, \ldots$$

It means first non-vanishing derivative at $x = a$ is of odd order, where $f^{(n)}(x)$ is nth derivative of $f(x)$.

(2) A point $(a, f(a))$ is called point of inflection if

$$f'(a) = 0 \text{ or } \frac{1}{f'(a)} = 0 \text{ and } f''(a) = 0 \text{ or } \frac{1}{f''(a)} = 0$$

but

$$f^{(n)}(a) \neq 0 \quad ; \quad n = 3, 5, 7, \ldots$$

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$$f'(x) = x^2 + 4x - 12 = (x-2)(x+6)$$
$$f''(x) = 2x + 4 = 2(x+2)$$

At critical points

$$f'(x) = 0$$
$$x^2 + 4x - 12 = 0$$
$$(x-2)(x+6) = 0$$
$$x = -6,\ 2$$
$$f''(-6) = 2(-6) + 4 = -8 < 0$$
$$f''(2) = 2(2) + 4 = 8 > 0$$

Now,

$$f(-6) = \frac{(-6)^3}{3} + 2(-6)^2 - 12(-6) + 4 = 76$$
$$f(2) = \frac{2^3}{3} + 2(2)^2 - 12(2) + 4 = \frac{-28}{3} = -9\frac{1}{3}$$

Thus $(-6, 76)$ and $\left(2,\ -9\frac{1}{3}\right)$ are stationary points (turning points).

The points $(-6, 76)$ and $\left(2,\ -9\frac{1}{3}\right)$ are relative maximum and relative minimum points respectively.

Points of Inflection:

$$f''(x) = 0$$
$$2(x+2) = 0$$
$$x = 2$$

and

$$f(-2) = \frac{100}{3} = 33\frac{1}{3}$$

Since $f'''(x) = 2 \neq 0$, for all x.

The point $\left(-2, 33\frac{1}{3}\right)$ is the point of inflection.

RELATIVE EXTREMA

In this section we shall be concerned with relative minima and relative maxima, which are also called relative extrema.

Relative Maxima:

A function f has a relative maximum value at x_0, if there is an open interval about x_0 on which $f(x_0)$ is the largest value, it can be defined as $f(x_0) \geq f(x)$ for all x in the interval.

Relative Minima:

A function f has a relative minimum value at x_0, if there is an open interval about x_0 on which $f(x_0)$ is the smallest value, it can be defined as $f(x_0) \leq f(x)$ for all x in the interval.

Explanation:

To understand the concept of relative minimum and relative maximum values of a function f, consider the graph of the function over the interval (a, b) as shown in

Figure 7.36

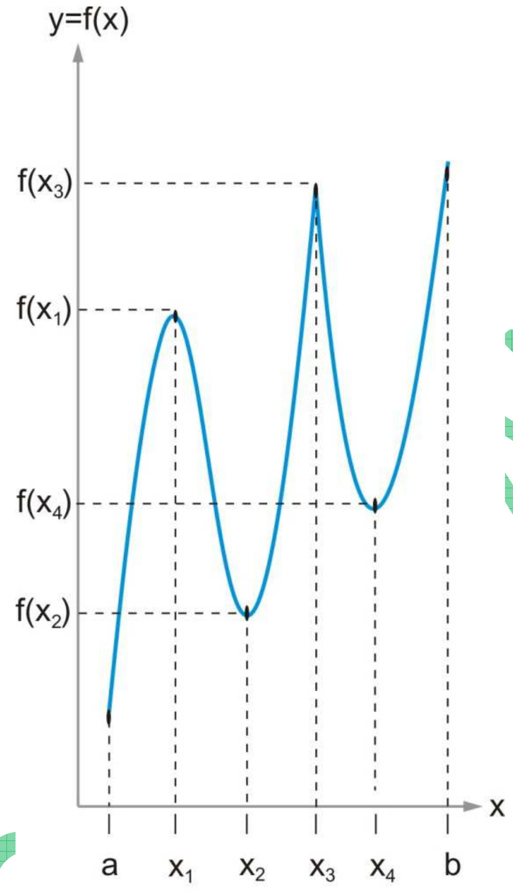

Figure 7.36

The graph of the function f, illustrates the values of f are relative maximum at x_1 and x_3 that are $f(x_1)$ and $f(x_3)$ because $f(x_1)$ and $f(x_3)$ are the largest values of f on open intervals about x_1 and x_3 respectively. Similarly $f(x_2)$ and $f(x_4)$ are the smallest values on the open interval about x_2 and x_4 respectively. So f is relative minima at x_2 and x_4.

The function f is not relative extrema at a and b, because there is not any open interval about a or b. But we can check absolute maximum or absolute minimum values at a and b.

The following derivative test can be used to determine a relative maximum or relative minimum of a function f.

FIRST DERIVATIVE TEST

Suppose that c is abscissa of a critical point and f is continuous on the open interval (a, b) where c belongs to (a, b).

(1) If $f'(x) < 0$ on (a, c) {on left side of c
and $f'(x) > 0$ on (c, b) {on right side of c
sign of derivative changes from negative to positive at c, so f has a relative minimum at c.

(2) If $f'(x) > 0$ on (a, c) {on left side of c}
and $f'(x) < 0$ on (c, b) {on right side of c}
sign of derivative changes from positive to negative at c, so f has a relative maximum at c.

(3) If $f'(x) < (0 \; or \; f'(x) > 0)$ on both the open intervals (a, c) and (c, b) then f does not have a relative extremum at c.

First Derivative test step by step:

Step 1: Find $f'(x)$.

Step 2: Locate all critical points by substituting

$$f'(x) = 0 \text{ and } f'(x) = \infty$$

Suppose that these equations give $x = c\,, d\,, k$.

Step 3: Write c, d and k in ascending order.

Step 4: Form interval like $(-\infty, c), (c, d), (d, k)$ and (k, ∞).

Step 5: Select a number from each interval and check the sign of $f'(x)$.

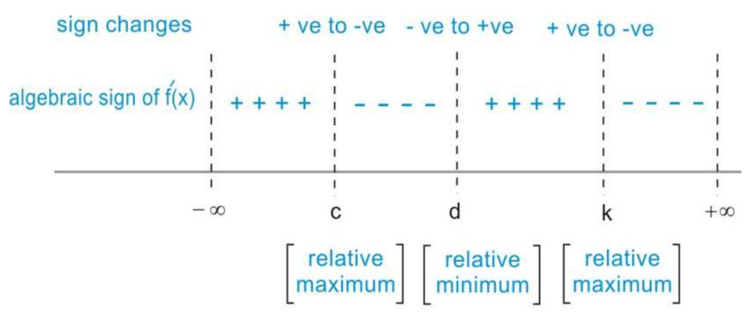

Figure 7.37

Note: The algebraic sign of $f'(x)$ cam not change over any of the intervals.

EXPLANATION GEOMETRICALLY:

The **figure 7.38** illustrates the critical points, tangents lines and their slopes at critical points and both sides of the critical points.

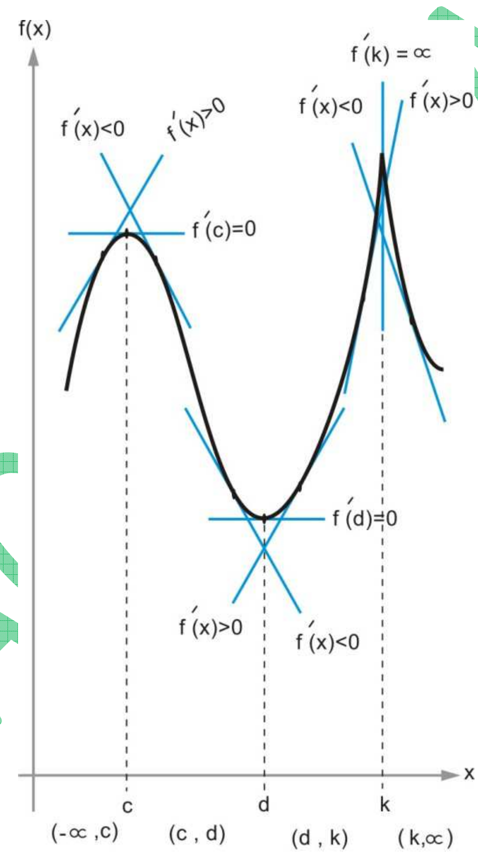

Figure 7.38

Example 7.10*:* Using first derivative test locate the relative extrema of the following functions.

Also find the relative extremum values.

(a) $f(x) = x^2 - 5x + 6$

(b) $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 14x$

Solution:

(a) $f(x) = x^2 - 5x + 6$

$f'(x) = 2x - 5$

If $f'(x) = 0$

$2x - 5 = 0$

$x = 2.5$

Only one critical point is at 2.5 which is stationary point. There are two intervals $(-\infty, 2.5)$ and $(2.5,\ \infty)$. Select a number from both the intervals and examine the algebraic sign of $f'(x)$. Selected numbers are 0 and 3.

$f(0) = -5 < 0$ on $(-\infty, 2.5)$

and $f(3) = 1 < 0$ on $(2.5,\ \infty)$

Since, the sign of derivative changes from negative to positive at $x = 2.5$, there is a relative minimum at 2.5.

Short Cut:

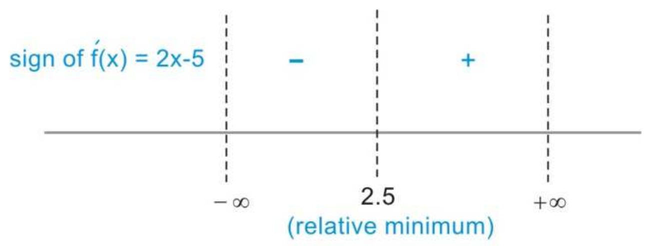

Figure 7.39

Sign changes from negative to positive at 2.5, there is relative minimum at 2.5.

Relative Minimum Values:

$$f(2.5) = -0.25$$

The minimum point is $(2.5, -0.25)$.

(b) $$f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 14x$$

$$f'(x) = x^2 + 5x - 14$$
$$= (x+7)(x-2)$$

If $$f'(x) = 0$$
$$(x+7)(x-2) = 0$$
$$x = -7 \ , \ x = 2$$

The critical points are stationary point at -7 and 2.

Examine the sign of $f'(x)$ in the intervals $(-\infty, -7), (-7, 2)$ and $(2, \infty)$

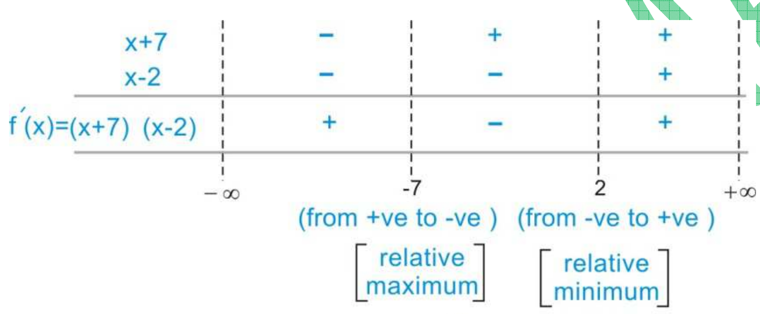

Figure 7.40

There is relative maximum at -7 and relative minimum at 2.

Relative Extreme Values:

$$f(-7) = 106\frac{1}{6} \text{ and } f(2) = -15\frac{1}{3}$$

The maximum and minimum points are $\left(-7, 106\frac{1}{6}\right)$ and $\left(2, -15\frac{1}{3}\right)$ respectively.

Example 7.11:

Using first derivative test determine the location (s) of critical points of the following functions and determine their nature. Also find the relative extreme values.

(a) $f(x) = (x-3)^{1/3}$

(b) $f(x) = (x-2)^3 + 3$

(c) $f(x) = 3(x-2)^{\frac{5}{3}} - 15(x-2)^{\frac{2}{3}} + 3$

Solution:

(a) $$f(x) = (x-3)^{1/3}$$

$$f'(x) = \frac{1}{3(x-3)^{2/3}}$$

If $f'(x) = 0$, there is not critical point.

If $f'(x) = \infty$ or $\dfrac{1}{f'(x)} = 0$

$$3(x-3)^{\frac{2}{3}} = 0$$

$$x = 3$$

The only critical point is at 3.

Examine the algebraic sign of $f'(x)$:

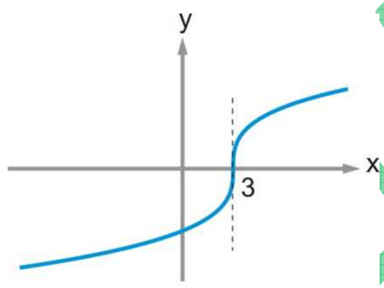

Figure 7.41

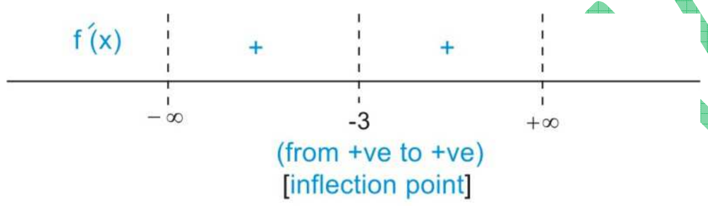

Figure 7.42

The critical point is inflection point at 3.

Since $f(3) = 0$, the inflection point is (3, 0).

(b) $$f(x) = (x-2)^3 + 3$$

$$f'(x) = 3(x-2)^2$$

If $f'(x) = 0 \quad \Rightarrow \quad x = 2$

Only critical point is at 2. {it is stationary point}
Examine the sign of $f'(x)$:

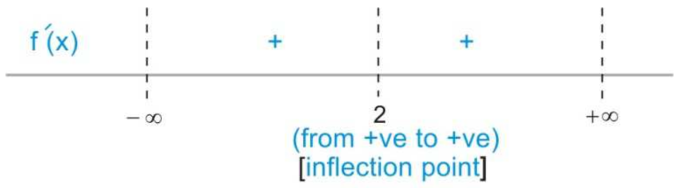

Figure 7.44

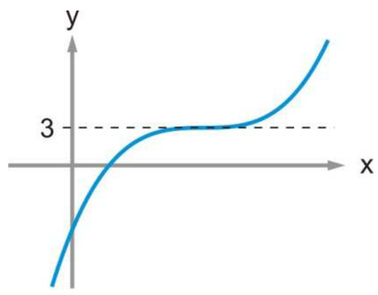

Figure 7.43

There is inflection point at 2.
Since $f(2) = 2$, the inflection point is (2, 2).

(c) $f(x) = 3(x-2)^{\frac{5}{3}} - 15(x-2)^{\frac{2}{3}} + 3$

$$f'(x) = 5(x-2)^{\frac{2}{3}} - 10(x-2)^{\frac{-1}{3}}$$

$$= 5(x-2)^{\frac{-1}{3}}\{(x-2) - 2\}$$

$$= \frac{5(x-4)}{(x-2)^{\frac{1}{3}}}$$

Case 1:

$$f'(x) = 0 \quad \Rightarrow \quad x = 4$$

Case 2:

$$f'(x) = \infty \text{ or } \frac{1}{f'(x)} = 0$$

$$\Rightarrow \quad x = 2$$

The critical points are at 2 and 4. The stationary point is at $x = 4$. Examine the algebraic sign of $f'(x)$.

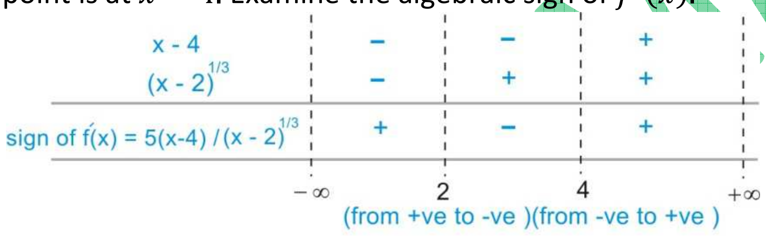

Figure 7.45

The relative maximum is at 2 and relative minimum is at 4.

Relative Extreme Values:

$f(2) = 3$ and $f(4) = -11.29$

The maximum and minimum points are (2, 3) and (4, -11.29) respectively.

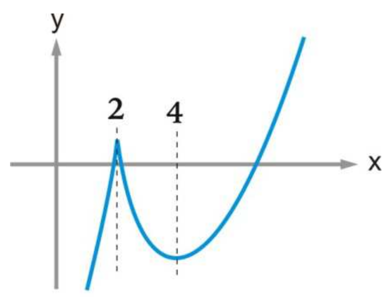

Figure 7.46

SECOND DERIVATIVE TEST

Suppose that $f'(c) > 0$ and f is twice differentiable at c.

(a) If $f''(c) > 0$, then f has a relative minimum at c.

(b) If $f''(c) < 0$, then f has a relative maximum at c.

(c) If $f''(c) = 0$, then the test is inclusive.

Try nth derivative test.

Example 7.12*:*

Using second derivative test find the relative maximum and relative minimum value of $f(x) = x^3 - 6x^2$.

Solution: $f(x) = x^3 - 6x^2$

$$f'(x) = 3x^2 - 12x$$

$$f''(x) = 6x - 12$$

Prime factors of $f'(x)$

$$f'(x) = 3x\,(x-4)$$

If $f'(x) = 0$ then $x = 0, x = 4$

Examine the algebraic sign of $f''(x)$ at 0 and 4.

$f''(0) = -12 < 0, f$ is relative maximum at $x = 0$.

$f''(4) = -12 > 0, f$ is relative minimum at $x = 4$.

<u>Relative Extreme Values:</u>

$f(0) = 0$ and $f(4) = -32$

nth DERIVATIVE TEST

Suppose that $f'(c) = 0$ and nth derivative of f exists.

If $f''(c) = f'''(c) = \ldots = f^{(n-1)}(c) = 0$

and $f^{(n)}(x) \neq 0$ {first non – vanishing derivative}

where n is the order of the derivative.

(a) n is even natural number.

If $f^{(n)}(c) < 0, f$ has a relative maximum at c.

If $f^{(n)}(c) > 0, f$ has a relative minimum at c.

(b) n is odd natural number.

The critical point is an inflection point.

Example 7.13:

Find the relative minimum and relative maximum values of

$$f(x) = \frac{9}{2}x^4 - \frac{3}{2}x^3 + 3$$

Solution:

$$f(x) = \frac{9}{2}x^4 - \frac{3}{2}x^3 + 3$$

$$f'(x) = 18x^3 - \frac{3}{2}x^3$$

$$f(x) = 54x^2 - 9x$$

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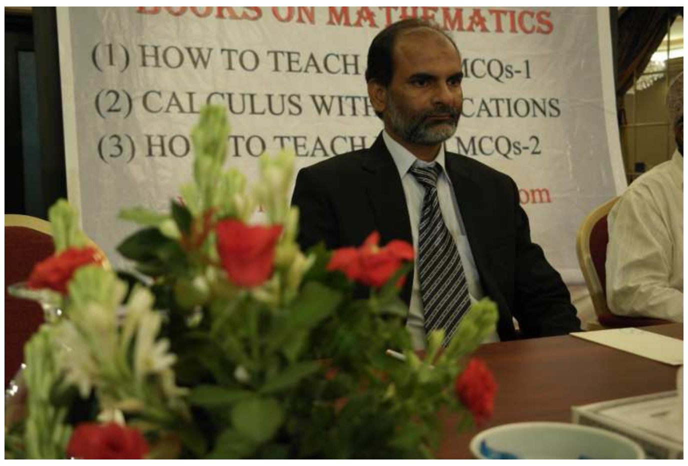

DIFFERENCE BETWEEN RELATIVE AND ABSOLUTE EXTREMA:

A relative maximum may also be absolute maxima, but it need not be so in every problem. If an absolute extrema occurs when x is inside (not at an end) of an interval of admissible values of x, then we also have a relative extrema. But if the absolute extrema occurs at one end of the interval it is not a relative extrema.

The **graphs 7.48 a,b,c** illustrates the difference between absolute extrema and relative extrema.

(a) absolute maximum = relative maximum = $f(x_1)$
absolute minimum = relative minimum = $f(x_2)$

(b) absolute maximum = $f(b)$
absolute minimum = relative minimum = $f(x_1)$
No relative maximum.

(c) absolute minimum = $f(a)$
relative maximum = $f(x_1)$
relative minimum = $f(x_2)$

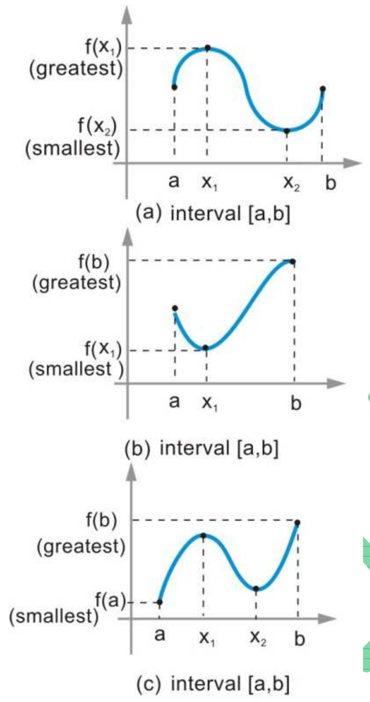

Figure 7.48 *a,b,c*

The **graphs 7.49 a,b,c,d** demonstrates that if the interval is infinite, then absolute maxima and absolute minima do not always occur.

(a) $f(x_1)$ is the absolute minimum no absolute maximum.

(b) No absolute minimum, no absolute maximum.
but relative minimum = $f(x_1)$
and relative maximum = $f(x_2)$

(c) $f(b)$ is the absolute maximum, no absolute minimum.
But relative maximum = $f(x_1)$
and relative minimum = $f(x_2)$.

(d) There is no absolute extrema and relative extrema.

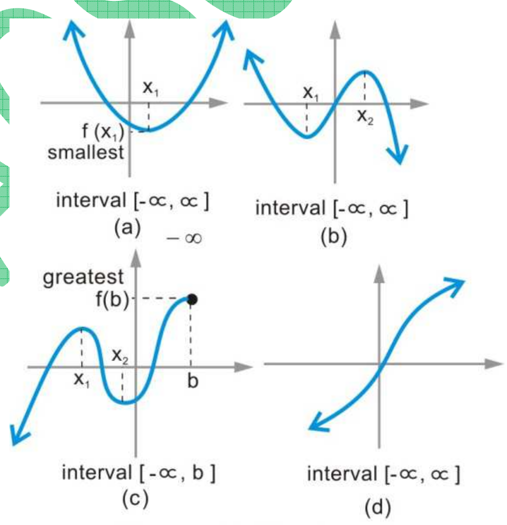

Figure 7.49 *a,b,c,d*

Example 7.14*:*

Find the values of relative maximum, relative minimum, absolute maximum and absolute minimum for the following function

$$f(x) = 2x^3 - 3x^2 - 36x + 12 \; ; \; x \in [-3, 2]$$

Solution:

$$f(x) = 2x^3 - 3x^2 - 36x + 12$$
$$f'(x) = 6x^2 - 6x - 36$$
$$= 6(x^2 - x - 6)$$
$$= 6(x+2)(x-3)$$

If $f'(x) = 0 \Rightarrow x = -2 \; and \; x = 3$

Since -2 belongs to $[-3, 2]$ but 3 does not.

So that the intervals are $(-\infty, -2)$ and $(-2, 2)$.

There is relative maximum value is at -2, that is

$f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 12 = 52$

So, relative maximum value = $f(-2) = 52$.

The values of $f(x)$ at the end points are

$f(-3) = 2(-3)^3 - 3(-3)^2 - 36(-3) + 12 = 147$

$f(2) = 2(2)^3 - 3(2)^2 - 36(2) + 12 = -56$

Since $f(-2) = 52, f(-3) = 147$ and $f(2) = -56$

Greatest value is $f(-3) = 147$

Smallest value is $f(2) = -56$

$\therefore$ Absolute maximum value = $f(-3) = 147$

Absolute minimum value = $f(2) = -56$

Example 7.15:

Business and Economics

There are two type of cost to export wheat. The fare cost and storage cost. The fare cost is 30 cents per kilogram and storage cost is a fraction of 7000 dollars and the mass of wheat in tonne. Find the cost function. Find the mass of wheat in kilograms for minimum cost in dollars.

Solution:

Suppose that x kilogram wheat is exported.

Per kg fare cost $= m = 30$ cent $= 0.3$ dollars

Storage cost $= 7000/(\frac{x}{1000})$ dollars$= \frac{7000000}{x}$ dollars

The cost function is

$$C(x) = 0.3x + \frac{7000000}{x} \quad \rightarrow (1)$$

To find minimum cost

$$C'(x) = 0.3 - \frac{7000000}{x^2} \quad \rightarrow (2)$$

$$C''(x) = \frac{14000000}{x^3} \quad \rightarrow (3)$$

For critical points

$$C'(x) = 0.3 - \frac{7000000}{x^2} = 0$$

$$x = 4830$$

Second derivative test

$$C''(4830) = \frac{14000000}{4830^3} > 0$$

$C(x)$ is minimum at $x = 4830$, so minimum cost is

$$C(4830) = 0.3 \times 4830 + \frac{7000000}{4830}$$

$$= 2898 \text{ dollars}$$

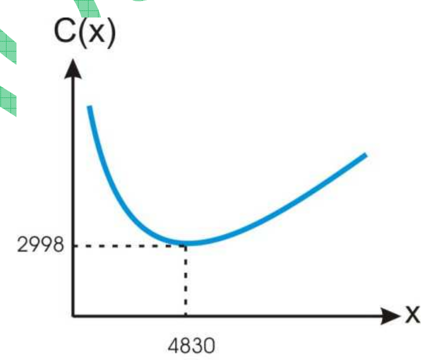

Figure 7.50A

Example 7.16:

A man has a right triangular tin sheet LMN as shown in the figure 2.25, where $LM = 8\,ft$ and $MN = 6\,ft$. If he wants to cut a rectangle of maximum area of dimensions x and y. Find area of rectangle as a function of x.

(a) Find the value of x for a rectangle of maximum area.

Solution:

Area of rectangle is

$$A = xy \qquad \rightarrow (1)$$

Since LMN and BCN are similar right triangle.

Therefore,

$$\frac{BC}{LM} = \frac{CN}{MN}$$

$$\frac{y}{8} = \frac{6-x}{6}$$

$$y = \frac{4}{3}(6-x) \qquad \rightarrow (2)$$

Putting the value of y in (1), we have

$$A(x) = \frac{4x}{3}(6-x)$$

$$A(x) = 8x - \frac{4}{3}x^2$$

is the area of rectangle as a function of x.

(a) To find maximum value of A

$$A'(x) = 8 - \frac{8}{3}x \qquad \rightarrow (3)$$

$$A''(x) = -\frac{8}{3} \qquad \rightarrow (4)$$

For critical points

$$A'(x) = 8 - \frac{8}{3}x = 0$$

$$x = 3$$

second derivative test

$$A''(3) = -\frac{8}{3} < 0$$

Area is maximum at $x = 3$.

Maximum area is

$$A(3) = 8 \times 3 - \frac{4}{3}(3)^2$$

$$= 12$$

Maximum area of the rectangle is 12 sq. ft at $x = 3$ ft.

Physical Science:

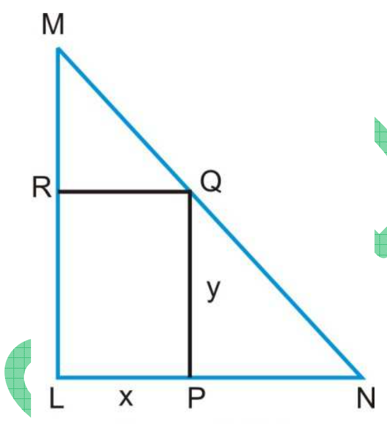

Figure 7.50B

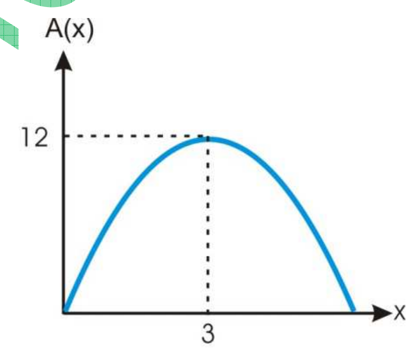

Figure 7.50C

Example 7.17: **Business and Economics**

The total cost and total revenue functions of q unit are

$$R(q) = -0.04q^2 + 20q$$
$$C(q) = 0.03q^2 + 2q + 200$$

(a) Find the total profit function and marginal profit function.

(b) Find the number of units where the profit is maximum.

(c) Find the maximum profit.

Solution:

(a) Total profit function is

$$P(q) = R(q) - C(q)$$
$$P(q) = -0.07q^2 + 18q - 200$$

Marginal profit function is

$$P'(q) = -0.14q + 18$$

(b) For critical points

$$P'(q) = 0$$
$$q = 129$$

The second derivative of $P(q)$ is

$$P''(q) = -0.14$$

Second derivative test

$$P''(129) = -0.14 < 0$$

Therefore, profit is maximum when the number of units are 129.

(c) Maximum profit:

$$P(129) = 957.13 \text{ dollars}$$

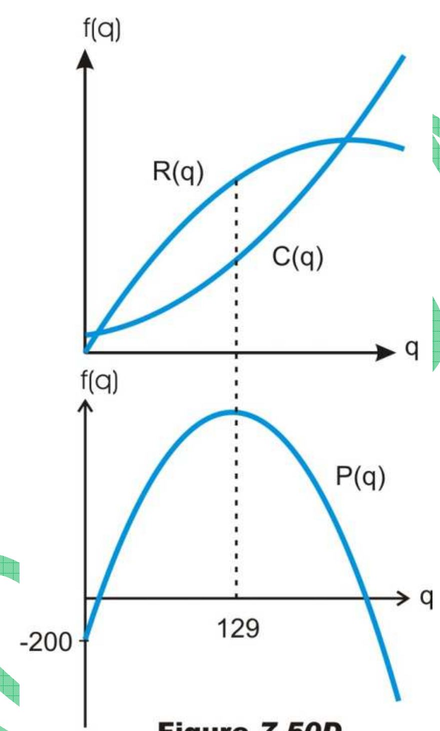

Figure 7.50D

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