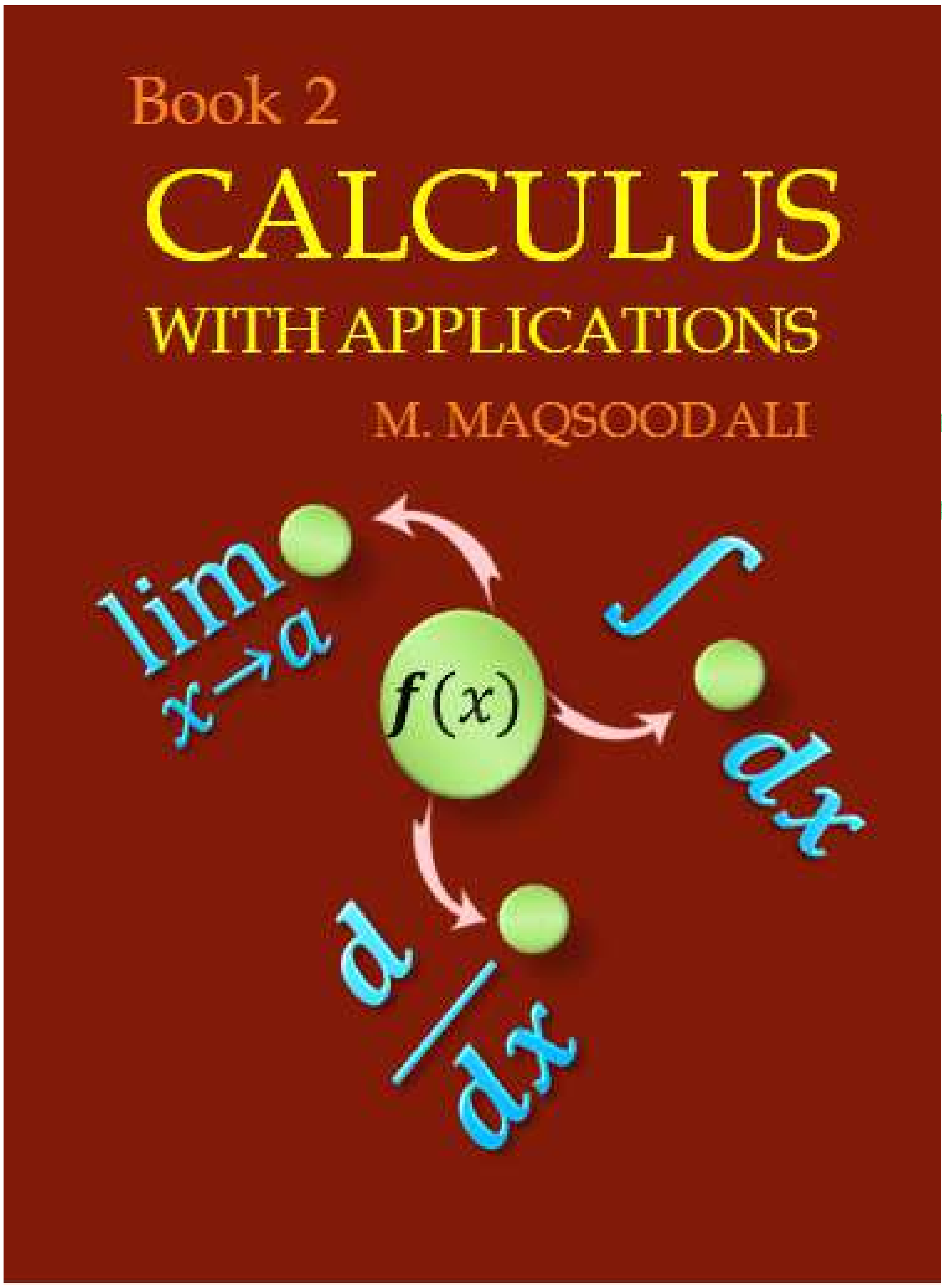
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
lim
x→a
f(x)
∫
dx
d
dx

CONTINUITY, DERIVATIVE AND DIFFERNTIATION

Theorem A-4:

(i) Every finitely derivable function is continuous.

(ii) Taking a suitable example prove that converse is not necessary true.

Proof:

Suppose that f is a finitely derivable function at $x = a$, so that

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \text{finite real number}$$

$$\text{Since}\quad f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$$

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right](x-a)$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a)$$

$$= f'(a).0 = 0$$

$$\Rightarrow \quad \lim_{x\to a} f(x)-f(a) = 0$$

$$\Rightarrow \quad \lim_{x\to a} f(x) = f(a)$$

so that f is continuous at $x = a$.

(ii) Let f be a modulus function.

$$f(x) = |x|$$

it is defined as

$$f(x) = |x| = \begin{cases} -x & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ x & \text{for} \quad x > 0 \end{cases}$$

We know that f is continuous at $x = 0$

Now we prove that f is not derivable at $x = 0$.

We have to prove that

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

does not exist.

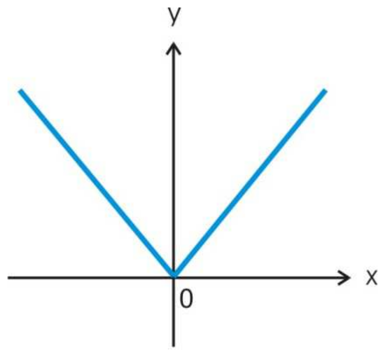

Figure 7.9

Left hand limit at 0:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{|x|-0}{x-0}$$

$$= \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = -1$$

Right hand limit at 0:

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|-0}{x-0}$$

$$=\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=1$$

$$\Rightarrow \quad \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}\neq \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$$

so that $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist.

Hence f is not derivable at $x=0$

Example 7.8:

Discuss derivability of the function $f(x)=\sin x$ at $x=0$.

Solutiion:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\sin x-\sin 0}{x-0}$$

$$=\lim_{x\to 0}\frac{\sin x-0}{x}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}=1$$

Hence f is derivable at $x=0$

DIFFERENTIABLE FUNCTIONS:

A function f is said to be differentiable at $x=x_0$, if there exists a relation of the form.

$$f(x_0+h)-f(x_0)=A.h+\varepsilon\, h$$

$\varepsilon\to 0$ as $h\to 0$

and A depends on x_0.

or

A function f is said to be differentiable at $x=x_0$, if there exists a relation of the form.

$$f(x+\Delta x)-f(x)=A.\Delta x+\varepsilon.\Delta x$$

or $\quad \Delta y=A.\Delta x+\varepsilon.\Delta x$

$\varepsilon\to 0$ as $\Delta x\to 0$

and A is a function of x.

DIFFERENTIAL:

$y=f(x)$ is a function and

$$dy=\frac{dy}{dx}\,\Delta x$$

dy is called differential of y.

COEFFICIENT OF DIFFERENTIAL:

The differential of $y = f(x)$ is defined as

$$dy = \frac{dy}{dx}\,\Delta x$$

$\frac{dy}{dx}$ is called coefficient of differential.

Differentiation and Derivative:

A function $f(x)$ is said to be differentiable at $x = x_0$ if there exists a relation of the form

$$f(x_0 + h) - f(x_0) = Ah + \varepsilon h$$

$\varepsilon \to 0$ as $h \to 0$

The relation can be written as

$$\frac{f(x_0 + h) - f(x_0)}{h} = A + \varepsilon$$

Taking $h \to 0$ ' so $\varepsilon \to 0$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = A$$

$$\Rightarrow \quad f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = A$$

The result is derivative. So differentiation is the process of finding the derivative of a function of one variable. We can say that derivability and differentiability are same for a function of one variable.

EXERCISE

Show that the following functions are continuous and derivable at the indicated point.

(1) $f(x) = \cos x$ at $x = 0$

(2) $f(x) = x^2 + 5x + 2$ at $x = 2$

(3) $f(x) = \sqrt{x} - 1$ at $x = 1$

(4) Show that $f(x) = \sin x$ is continuous and derivable for all $x \in \mathbb{R}$.