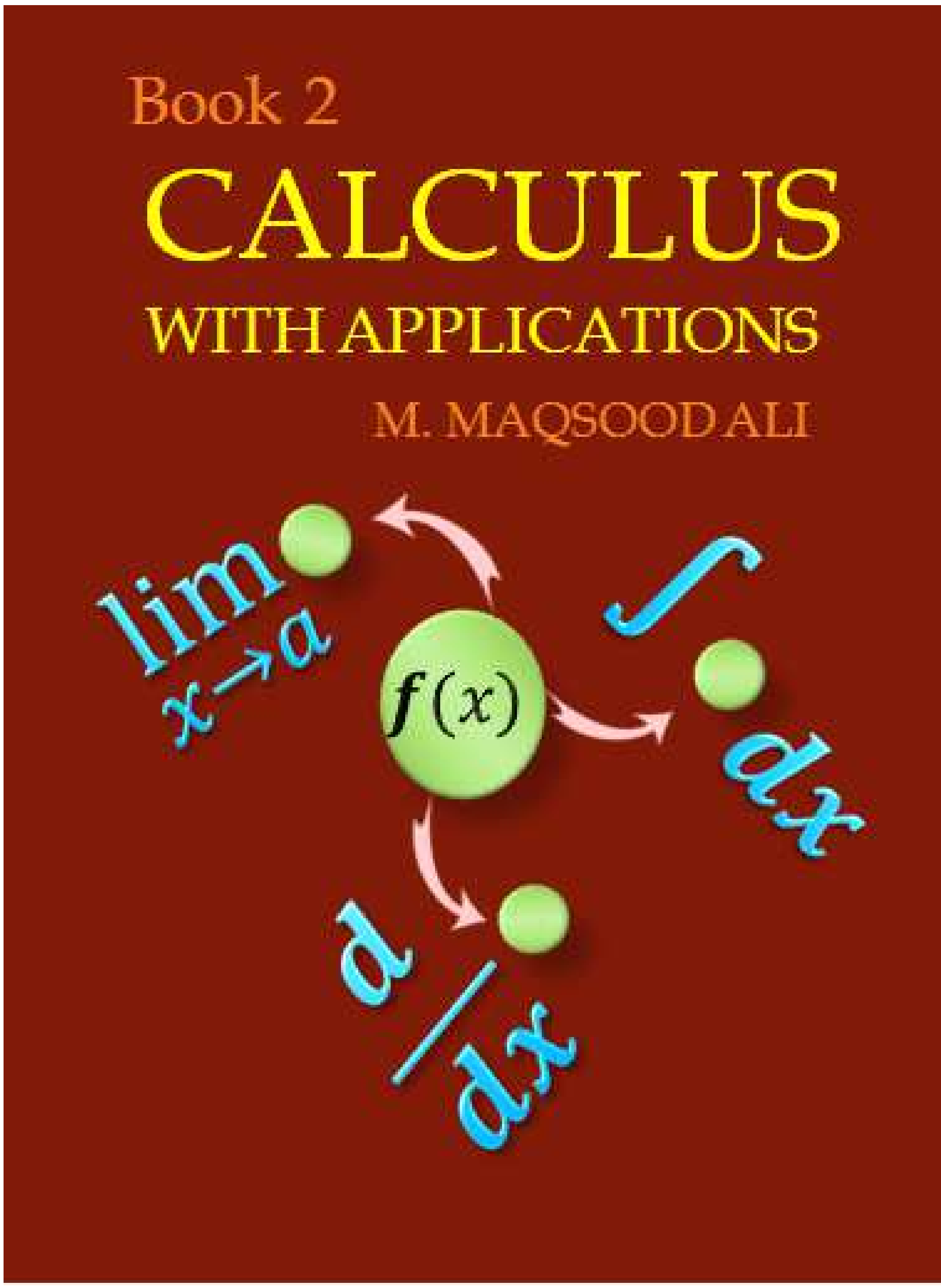
Book 2
CALCULUS
WITH APPLICATIONS
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lim
x→a
f(x)
dx
d
dx

LEIBNITZ THEOREM

Statement:

If f and g are functions of a variable x, then nth derivative of $(f.g)$ can be defined as

$$(f.g)^{(n)} = f^{(n)}\,g + {}^{n}C_1\,f^{(n-1)}\,g^{(1)} + {}^{n}C_2\,f^{(n-2)}\,g^{(2)} + \cdots + {}^{n}C_1\,f^{(1)}\,g^{(n-1)} + fg^{(n)}$$

where $(f.g)^{(n)} = D^n\,(f.g)$, $D^n = \dfrac{d^n}{dx^n}$ and ${}^{n}C_r = \dfrac{n!}{(n-r)!\,r!}$

Proof:

Suppose that $P(n)$ represents the proposition

$(f.g)^{(n)}$

$$= f^{(n)}\,g + {}^{n}C_1\,f^{(n-1)}\,g^{(1)} + {}^{n}C_2\,f^{(n-2)}\,g^{(2)} + \cdots + {}^{n}C_{n-1}\,f^{(1)}\,g^{(n-1)} + fg^{(n)} \rightarrow (1)$$

For $n = 1$:

First derivative of $(f.g)$

$$(f.g)^{(1)} = D\,(f.g)$$

$$= f^{(1)}g + fg^{(1)} \qquad \rightarrow (2)$$

On substituting $n = 1$ in (1), we get

$$(f.g)^{(1)} = f^{(1)}g + fg^{(1)}$$

which is same as (2)

$P(n)$ is true for $n = 1$.

For $n = 2$:

Second derivative of $(f.g)$

$$(f.g)^{(2)} = D^2\,(f.g) = D\,\{D\,(f.g)\}$$

$$= D\,(f^{(1)}g + fg^{(1)})$$

$$= f^{(2)}g + f^{(1)}\,g^{(1)} + f^{(1)}\,g^{(1)} + fg^{(2)}$$

$$= f^{(2)}g + 2f^{(1)}\,g^{(1)} + fg^{(2)} \quad \rightarrow (3)$$

Substituting $n = 2$ in (1), we get

$$(f.g)^{(2)} = f^{(2)}g + {}^{n}C_1\, f^{(1)}\, g^{(1)} + fg^{(2)}$$

$$= f^{(2)}g + 2f^{(1)}\, g^{(1)} + fg^{(2)}$$

which is same as (3).

For $n = k$:

Suppose that $P(n)$ is true for $n = k$.

$$(f.g)^{(k)} = f^{(k)}g + {}^{k}C_1\, f^{(k-1)}\, g^{(1)} + {}^{k}C_2\, f^{(k-2)}\, g^{(2)} + \cdots + {}^{k}C_{k-1}\, f^{(1)}\, g^{(k-1)} + fg^{(k)}$$

For $n = k + 1$:

We prove that $P(n)$ is true for $n = k + 1$

$$(f.g)^{(k+1)} = D^{k+1}\,(f.g) = D\,\{D^{k}\,(f.g)\}$$

$$= D\,\{f^{(k)}g + {}^{k}C_1\, f^{(k-1)}\, g^{(1)} + {}^{k}C_2\, f^{(k-2)}\, g^{(2)} + \cdots + {}^{k}C_{k-1}\, f^{(1)}\, g^{(k-1)} + fg^{(k)}\}$$

$$= D\,(f^{(k)}g) + {}^{k}C_1\, D\big(f^{(k-1)}\, g^{(1)}\big) + {}^{k}C_2\, D\big(f^{(k-2)}\, g^{(2)}\big) + \cdots + {}^{k}C_{k-1}\, D\big(f^{(1)}\, g^{(k-1)}\big)$$
$$+ D(fg^{(k)})$$

$$= f^{(k+1)}g + f^{(k)}\, g^{(1)} + {}^{k}C_1\, f^{(k)}\, g^{(1)} + {}^{k}C_1\, f^{(k-1)}\, g^{(2)} + {}^{k}C_2 f^{(k-1)}\, g^{(2)} + {}^{k}C_2 f^{(k-2)}\, g^{(3)}$$
$$+ \cdots + {}^{k}C_{k-1} f^{(2)}\, g^{(k-1)} + {}^{k}C_{k-1} f^{(1)} g^{(k)} + f^{(1)} g^{(k)} + fg^{(k+1)}$$

$$= f^{(k+1)}g + ({}^{k}C_0 + {}^{k}C_1) f^{(k)} g^{(1)} + ({}^{k}C_1 + {}^{k}C_2) f^{(k-1)} g^{(2)} + \cdots$$
$$+ ({}^{k}C_{k-1} + {}^{k}C_k) f^{(1)} g^{(k)} + fg^{(k+1)}$$

since $\quad {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r$

Hence

$(f.g)^{(k+1)}$

$$= f^{(k+1)}g + {}^{k+1}C_1 f^{(k+1-1)} g^{(1)} + {}^{k+1}C_2 f^{(k+1-2)} g^{(2)} + \cdots + {}^{k+1}C_k f^{(1)} g^{(k+1-1)} + fg^{(k+1)}$$

$P(n)$ is true for $n = k + 1$

Hence $P(n)$ is true for all positive integral values of n.

Example 7.4:

Find the fourth derivative of $x^4 sinx$ with respect to $x, using$ Leibnitz theorem

Solution:

$f(x) = \sin x$, $g(x) = x^4$

$f^{(1)}(x) = \cos x$, $g^{(1)}(x) = 4x^3$

$f^{(2)}(x) = -\sin x$, $g^{(2)}(x) = 12x^2$

$f^{(3)}(x) = -\cos x$, $g^{(3)}(x) = 24x$

$f^{(4)}(x) = \sin x$, $g^{(4)}(x) = 24$

By Leibnitz theorem

$$(f.g)^{(4)} = f^{(4)}g + {}^4C_1 f^{(3)}g^{(1)} + {}^4C_2 f^{(2)}g^{(2)} + {}^4C_3 f^{(1)}g^{(3)} + fg^{(4)}$$
$$= x^4 \sin x - 16\,x^3 \cos x - 72\,x^2 \sin x + 96\,x \cos x + 24 \sin x.$$

Example 7.5:

Find nth derivative of $x^4\ sin\ x$.

Solution:

$f(x) = \sin x$, $g(x) = x^4$

$f^{(1)}(x) = \cos x$, $g^{(1)}(x) = 4x^3$

$f^{(2)}(x) = -\sin x$, $g^{(2)}(x) = 12x^2$

$f^{(3)}(x) = -\cos x$, $g^{(3)}(x) = 24x$

$f^{(4)}(x) = \sin x$, $g^{(4)}(x) = 24$

$f^{(n)}(x) = \sin(x + \frac{n\pi}{2})$

By Leibnitz theorem

$$(f.g)^{(n)} = f^{(n)}g + {}^nC_1 f^{(n-1)}g^{(1)} + {}^nC_2 f^{(n-2)}g^{(2)} + {}^nC_3 f^{(n-3)}g^{(3)} + {}^nC_4 f^{(n-4)}g^{(4)}$$

{The fifth and higher derivative of g is zero}

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Example 7.7:

If $y = a\ cos\ (log\ x)$, show that

$$x^2\ y^{(n+2)} + (2n+1)\ xy^{(n+1)} +\ (n^2 +\ 1)\ y^{(n)} = 0$$

Solution:

$$y = a\cos\ (log\ \ x)$$

Differentiate with respect to x

$$y^{(1)} = -a\ \frac{1}{x}\sin\ (\log x)$$

$$xy^{(1)} = -a\ \sin\ (\log x)$$

Again differentiate with respect to x

$$y^{(1)} + xy^{(2)} = -a\ \frac{1}{x}\ \cos\ (\log x)$$

$$y^{(1)} + xy^{(2)}\ = -\ \frac{1}{x}\ (\mathrm{a}\cos\ (\log x))$$

$$y^{(1)} + xy^{(2)}\ = -\frac{1}{x}\ \mathrm{y}$$

$$x^2y^{(2)} +\ xy^{(1)} +\ y = 0 \qquad \rightarrow (1)$$

By Leibnitz theorem

$$(f.g)^{(n)} = f^{(n)}g + {}^{n}C_1f^{(n-1)}g^{(1)} + {}^{n}C_2f^{(n-2)}g^{(2)} + \cdots +\ fg^{(n)}$$

nth derivative of equation (1).

$$(x^2y^{(2)})^{(n)} +\ (xy^{(1)})^{(n)} +\ (y)^{(n)} = 0$$

Applying Leibnitz theorem, we get

$$\left[y^{(n+2)}\ x^2 + {}^{n}C_1\ y^{(n+1)}\ (2x) + {}^{n}C_2\ y^{(n)}\ (2)\right] +\ \left[y^{(n+1)}\ x + {}^{n}C_1\ y^{(n)}\ (1)\right] +\ y^{(n)} = 0$$

$$x^2y^{(n+2)} +\ 2nxy^{(n+1)} +\ (n^2 -\ n)\ y^{(n)} +\ xy^{(n+1)} +\ ny^{(n)} +\ y^{(n)} = 0$$

$$x^2y^{(n+2)} +\ (2n+1)xy^{(n+1)}xy^{(n+1)} + (n^2 -\ n + n + 1)y^{(n)} = 0$$

$$x^2y^{(n+2)} +\ (2n+1)xy^{(n+1)} + (n^2 -\ n)\ y^{(n)} = 0$$

EXERCISE

Find the 4th derivative of the following functions.

(1) $F(x) = x^4 e^{6x}$

(2) $F(x) = \sin x \cos x$

(3) $F(x) = (x-1)^7 \cos x$

Find the 6th derivative of the following functions.

(4) $F(x) = x^2 \sin x$

(5) $F(x) = x^4 e^{ax}$

(6) $F(x) = (x-5)^7 \cos 5x$

Find the nth differential coefficient of the following functions.

(7) $F(x) = x^3 \cos x$

(8) $F(x) = x^2 \log x$

(9) show that

$$x^2 y^{(n+2)} + (2n+1)\, xy^{(n+1)} + (n^2+1)\, y^{(n)} = 0$$

$$if\ y = \sin(\log x) + \cos(log x)$$

(10) $If\ y = ae^{bx2+1}$, then show that

$$y^{(n+2)} - 2bxy^{(n+1)} - 2b\,(n+1)y^{(n)} = 0$$

(11) $If\ y = ae^{logx+b}$, then show that

$$x^2 y^{(n+2)} + (2n+1)\, xy^{(n+1)} + (n^2+1)\, y^{(n)} = 0$$