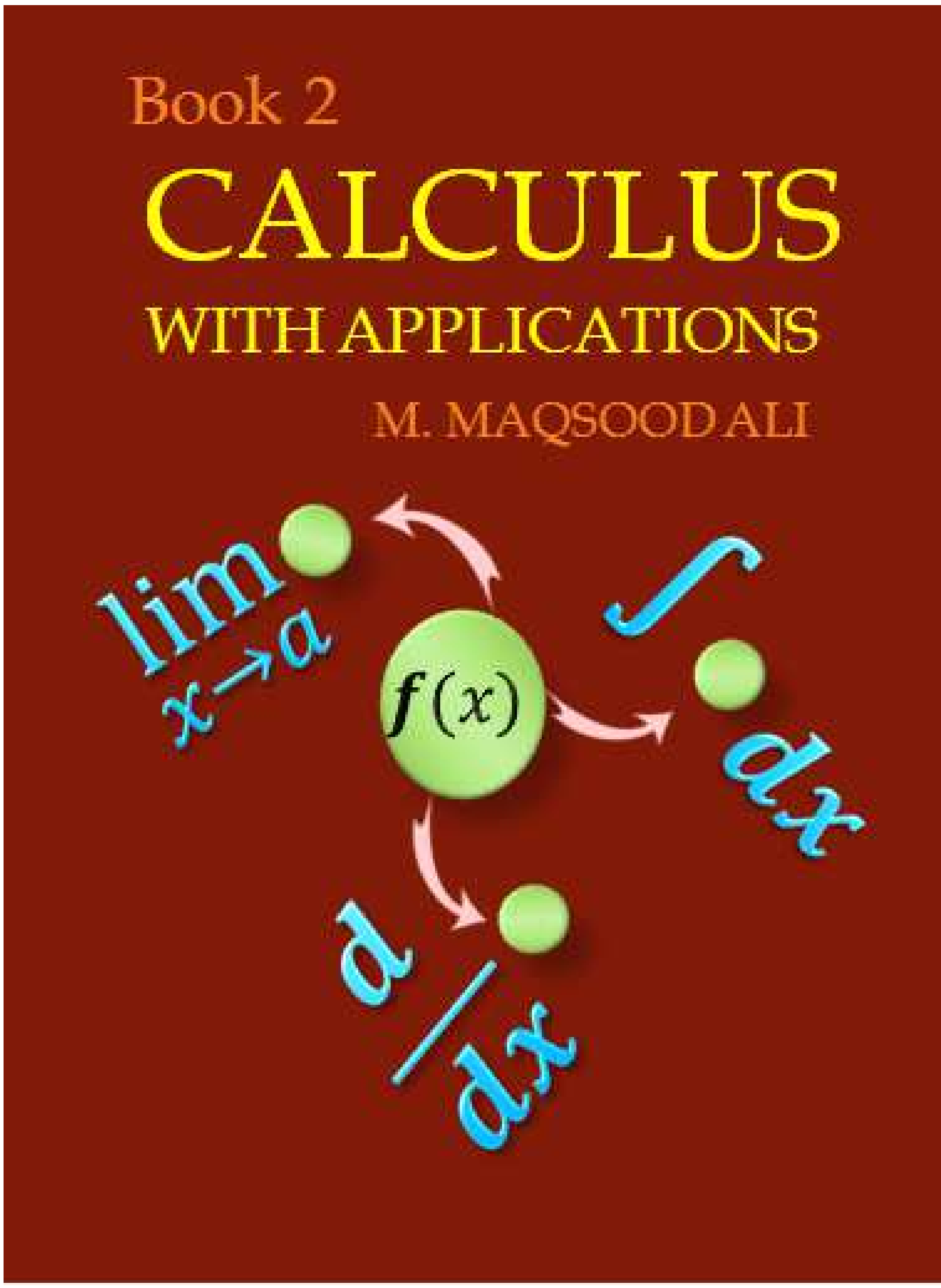
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
lim
x→a
f(x)
∫
dx
d
dx

Chapter 7

DERIVATIVES

RATE OF CHANGE IN A STRAIGHT LINE:

The graph of a linear function

$$f(x) = ax + b$$

is a straight line, as shown in the

figure 7.1

To find the rate of change in y with respect to x, select two points $A(x_1, y_1)$ and $B(x_2, y_2)$

$$m = \text{rate of change of } y \text{ w.r.t. } x = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Select other two points lie on the straight line to find rate of change of y w.r.t. x, the rate of change will be same. So that rate of change of a straight line is constant. This rate of change is called slope of the line.

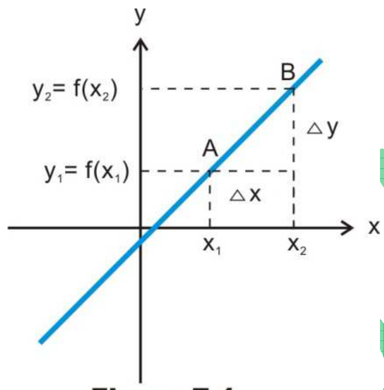

Figure 7.1

RATE OF CHANGE IN A CURVE

Suppose that $g(x)$ is a non-linear function. Graph of this function is shown in the **figure 7.2**.

P and Q are two points on the curve. A line segment joining P and Q is called secant and the line which passes through these two points is called secant line. If m_1 is the slope of secant PQ, than

$$\text{Slope of } PQ = m_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 7.3

Now select other three points R, S and T on the curve. According to the figure 7.3 slope of $\overline{PS}$is less than the slope of $\overline{PQ}$ and slope of $\overline{PR}$ is zero because $\overline{PR}$ is parallel to x-axis and the slope of PT is negative.

Above discussion tells that the slopes of two different secants drawn from P are not same. So what should be the procedure to find the rate of change at a point P, because different secants drawn from P such that PQ, PR, PS and PT have different slopes.

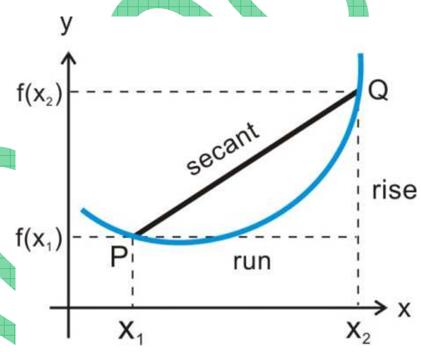

Figure 7.2

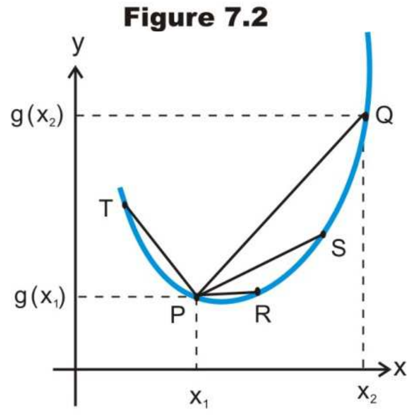

Figure 7.3

So what should do to find the rate of change at P? It is possible only when we select a point to draw a secant which is nearer to P. Suppose that $(x_0, f(x_0))$ are the coordinate of P and $(x_0 + h, f(x_0 + h))$ are the coordinates of another point on the curve. A secant line is drawn passes through these two points as shown in the **figure 7.4**. If $h \to 0$, the slope of secant line becomes slope of tangent line. A tangent line is a straight line which touches the curve at a point, **figure 7.5**. So that the rate of change at P is the slope of the tangent at P, which is given below.

$$\lim_{h\to 0}\frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}$$

$$\lim_{h\to 0}\frac{f(x_0 + h) - f(x_0)}{h}$$

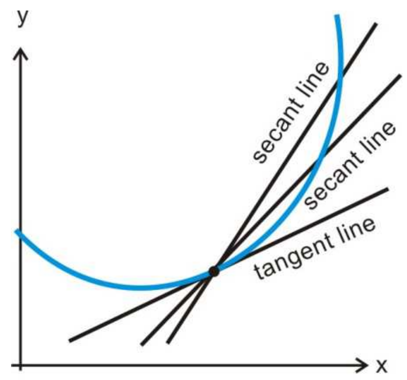

Figure 7.4

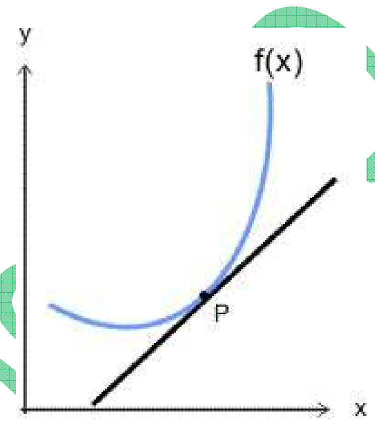

Figure 7.5

The value of this instantaneous rate of change of y with respect to x defined as $f'(x)$, {read f' as "f prime"}, so

$$f'(x) = \lim_{h\to 0}\frac{f(x_0 + h) - f(x_0)}{h}$$

f' is called the derivative of f.

AVERAGE AND INSTANTANEOUS RATE OF CHANGE

Slope of the secant joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is the average rate of change of $f(x)$ with respect to x over the interval(x_0, x_1).

$$m_{sec} = \frac{rise}{run} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Average rate of change of f w.r.t. x

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Slope of the tangent at point $(x_0, f(x_0))$ is instantaneous rate of change of $f(x)$ with respect to x at x_0.

$$m_{tan} = \lim_{x_1\to x_0}\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Instantaneous rate of change at x_0

$$= \lim_{x_1\to x_0}\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

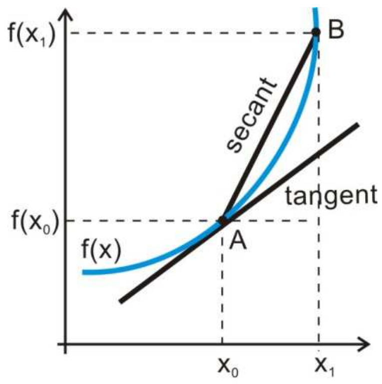

Figure 7.6

Figure 7.6

DERIVATIVES

The instantaneous rate of change of f at x or slope of the tangent of a curve $f(x)$ at x is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If this limit exist, the value of the limit is called derivative of $f(x)$ with respect to x and written as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The function f' is called derivative of function f, read as "f prime" and $f'(x)$, "f prime x".

DERIVABLE FUNCTIONS

A function is said to be derivable at $x = x_0 \in D_f$, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exist as a finite definite quantity.

As the signs $+$, $-$, $\times$ and $\div$ are used for addition, subtraction, multiplication and division respectively. Similarly the notation $\frac{d}{dx}$ or D_x (simply D) are used for derivative with respect to x.

Example 7.1:

Find the derivative by first principle of the following function

$f(x) = x^n$, n is a real number.

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}, \quad \textbf{figure 7.7}$$

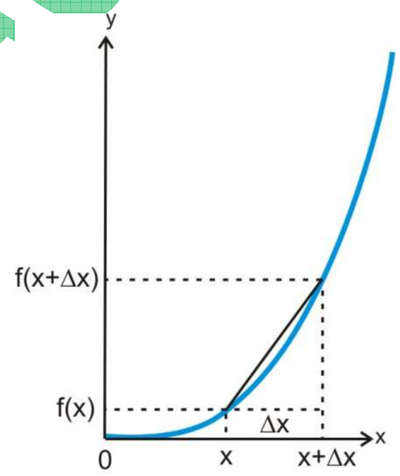

Figure 7.7

$$= \lim_{\Delta x \to 0} \frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x}$$

Using binomial theorem,

$$= \lim_{\Delta x \to 0} \frac{x^n \left\{ \begin{array}{l} 1 + n \,.\, \frac{\Delta x}{x} + \frac{n(n-1)}{2!} \,.\, \frac{\Delta x^2}{x^2} \\ + \frac{n(n-1)(n-2)}{3!} \,.\, \frac{\Delta x^3}{x^3} + \cdots \end{array} \right\} - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + n\,.\,\Delta x\,.\,x^{n-1} + \frac{n(n-1)}{2!}\Delta x\,.\,x^{n-2} + \frac{n(n-1)(n-2)}{3!}\Delta x^3\,.\,x^{n-3} + \ldots - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\ \{nx^{n-1} + \frac{n(n-1)}{2!}\Delta x\,.\,x^{n-2} + \frac{n(n-1)(n-2)}{3!}\Delta x^2\,.\,x^{n-3} + \ldots\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + \frac{n(n-1)}{2!}\Delta x.\ x^{n-2} + \frac{n(n-1)(n-2)}{3!}\Delta x^2\,.\,x^{n-3} + \ldots$$

$$= nx^{n-1} + 0 + 0 + \ldots$$
$$= nx^{n-1}$$

Example 7.2:

Find the derivative by first principle of the following functions $g(x) = sinx$.

Solution:

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{sin(x + \Delta x) - sin(x)}{\Delta x}, \quad \textbf{figure 7.8}$$

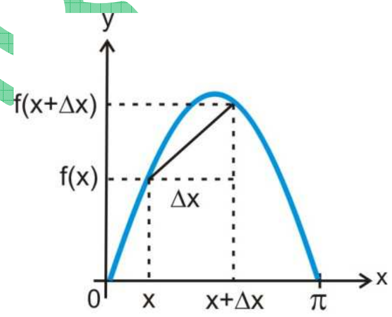

Figure 7.8

$$= \lim_{\Delta x \to 0} \frac{2\ \cos\frac{2x + \Delta x}{2}\ \sin(\frac{\Delta x}{2})}{\Delta x}$$

Multiplying and dividing the denominator (Δx) by 2.

$$= \lim_{\Delta x \to 0} \frac{2\ \cos\frac{2x + \Delta x}{2}\ \sin(\frac{\Delta x}{2})}{2\ .\ \frac{\Delta x}{2}}$$

$$= \lim_{\Delta x \to 0} \cos(\frac{2x + \Delta x}{2}).\ \lim_{\Delta x \to 0} \frac{\sin(\frac{\Delta x}{2})}{(\frac{\Delta x}{2})}$$

$$= cos\left(\frac{2x}{2}\right)(1)$$
$$= cosx$$

AUTHOR

M. MAQSOOD ALI

ASSISTANT PROFESSOR OF MATHEMATICS

(1) Sum Rule:

If u and v are derivable functions of x, then

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$
$$(u+v)'=u'+v'$$

Proof:

By the definition of derivative

$$\frac{d}{dx}\left[u(x)+v(x)\right]$$
$$=\lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$
$$=\lim_{h\to 0}\frac{u(x+h)-u(x)+v(x+h)-v(x)}{h}$$
$$=\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}+\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}$$
$$=\frac{d}{dx}u(x)+\frac{d}{dx}v(x)$$

(2) Subtraction Rule:

If u and v are derivable function of x, then

$$\frac{d}{dx}(\mathrm{u}-\mathrm{v})=\frac{du}{dx}-\frac{dv}{dx}$$
$$or \qquad (u-v)'=u'-v'$$

Proof:

By the definition of derivative

$$\frac{d}{dx}\left[u(x)-v(x)\right]$$
$$=\lim_{h\to 0}\frac{[u(x+h)-v(x+h)]-[u(x)-v(x)]}{h}$$
$$=\lim_{h\to 0}\frac{[u(x+h)-u(x)]-[v(x+h)-v(x)]}{h}$$
$$=\lim_{h\to 0}\frac{[u(x+h)-u(x)]}{h}-\lim_{h\to 0}\frac{[v(x+h)-v(x)]}{h}$$
$$=\frac{d}{dx}u(x)-\frac{d}{dx}v(x)$$

AUTHOR

M. MAQSOOD ALI

ASSISTANT PROFESSOR OF MATHEMATICS

(4) Quotient Rule:

If u and v are derivable functions of x, and $v \neq 0$, then

$$\frac{d}{dx}[\frac{u}{v}] = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

or $$\left(\frac{u}{v}\right)' = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Proof:

According to the definition of derivative

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0} \frac{v(x) \cdot u(x+h) - u(x) \cdot v(x+h)}{h \cdot v(x+h) \cdot v(x)}$$

Adding and subtracting $v(x) \cdot u(x)$ in numerator.

$$= \lim_{h\to 0} \frac{v(x) \cdot u(x+h) - v(x) \cdot u(x) - u(x) \cdot v(x+h) + v(x) \cdot u(x)}{h \cdot v(x+h) \cdot v(x)}$$

$$= \lim_{h\to 0} [\frac{1}{v(x+h) \cdot v(x)}] [\, v(x) \lim_{h\to 0} \{\frac{u(x+h) - u(x)}{h}\} - u(x) \lim_{h\to 0} \{\frac{v(x+h) - v(x)}{h}\}]$$

$$= \frac{1}{v(x) \cdot v(x)} [v(x) \frac{d}{dx} u(x) - u(x) \frac{d}{dx} v(x)]$$

$$= \frac{v(x) \frac{d}{dx} u(x) - u(x) \frac{d}{dx} v(x)}{v^2 (x)}$$