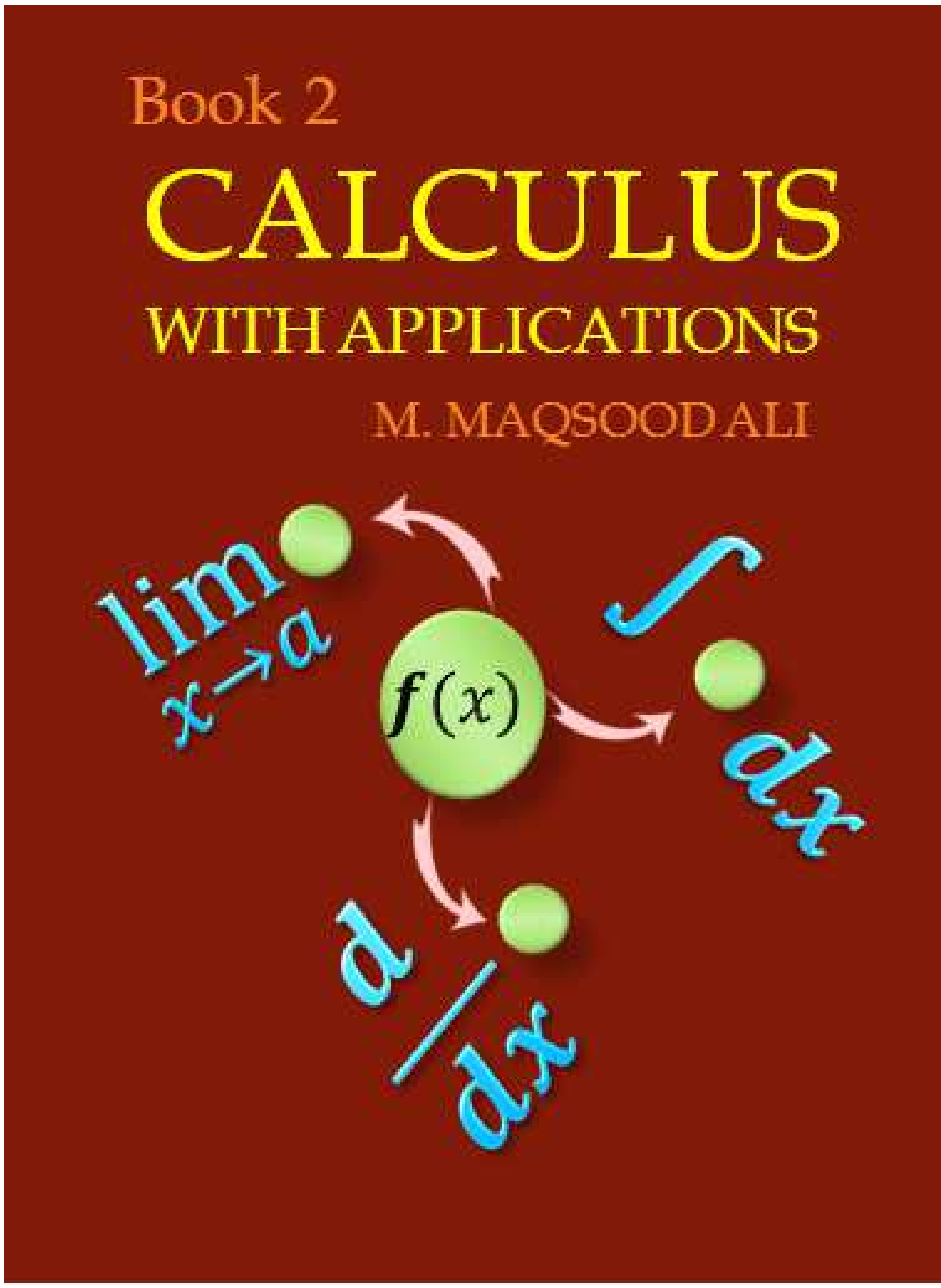

Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
lim
x→a
f(x)
∫
dx
d/dx

Chapter 6

CONTINUITY

. . . "if the function is continuous." . . .

It is not only a sentence but a necessary condition for almost all the theorems and formulas of Calculus. A function is continuous for all values in the domain of the function but discontinuous at only a point then the formulas of calculus will not exist for this function. When a function is formed , it is necessary to know about the function that either it is continuous or not in the domain of the function.

Definition 1:

A function $f(x)$ is continuous at $x = a$, if for a given positive real number ε there exits a corresponding real number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad for \quad |x - a| < \delta$$

Definition 2:

A function $f(x)$ is continuous at $x = a$, if the following conditions are satisfied.

(1) The value of $f(x)$ must exists at $x = a$

(2) $\lim_{x \to a} f(x)$ must exists

(3) $\lim_{x \to a} f(x) = f(a)$

Example 6.1:

Show that sin x is continuous for all values of $x \in \mathbb{R}$

Solution:

We discuss the continuity of f at a $\in \mathbb{R}$

$$|f(x) - f(a)| = |sin\, x - sin\, a|$$

$$= |2\, cos\, \frac{x + a}{2} . sin\, \frac{x - a}{2}|$$

$$= 2|cos\, \frac{x + a}{2}|.|sin\, \frac{x - a}{2}| \quad \rightarrow (1)$$

$$cos|\frac{x + a}{2}| \leq 1 \text{ and } |sin\, \frac{x - a}{2}| \leq |\frac{x - a}{2}|.$$

Substitute these values in (1)

$$|f(x) - f(a)| \leq 2(1)\, |\frac{x - a}{2}| = |x - a|$$

Let $|x - a| < \delta \; => |f(x) - f(a)| < \delta = \varepsilon$

$\because \quad |f(x) - f(a)| < \varepsilon$ for $|x - a| < \delta$

So f is continuous for all values of $x \in \mathbb{R}$

TYPES OF DISCONTINUOUS FUNCTIONS

Following are the three types of discontinuous functions.

The following functions are discontinuous at c.

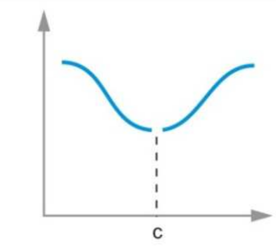
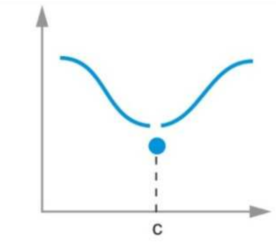
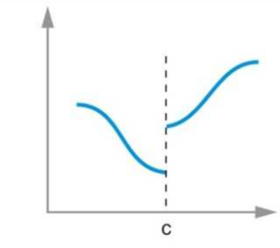
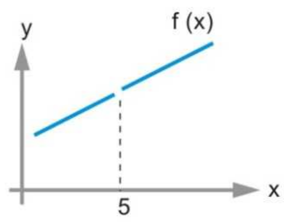
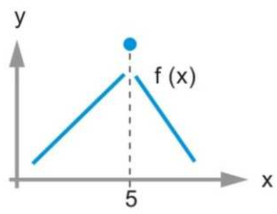
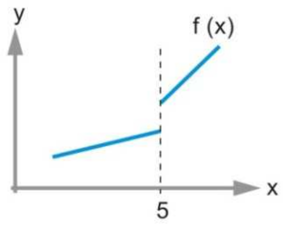
GAP	BREAK	JUMP
		
Examples: Following functions f(x) are discontinuous at 5.		
$f(x) = \frac{x^2 - 25}{x - 5}$	$f(x) = \begin{cases} 2x & , x < 5 \\ 12 & , x = 5 \\ -3x + 25 & , x > 5 \end{cases}$	$f(x) = \begin{cases} x + 3 & , x < 5 \\ 4x - 8 & , x \geq 5 \end{cases}$
		

Figure 6.1

Explanation:

If you draw a $f(x)$ from a to b without lifting the pen from the paper, as shown in the figure, then $f(x)$ is continuous on $[a, b]$.

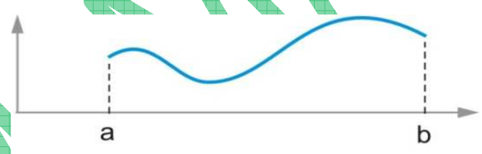


Figure 6.2

Confusion:

Can we discuss the continuity of the following functions?

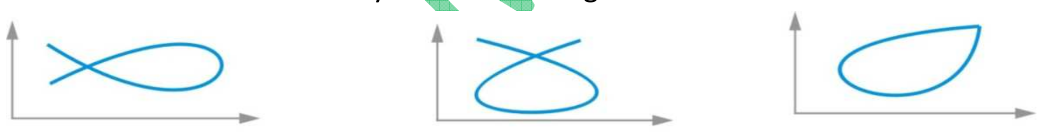


Figure 6.3

By vertical line test the above curves are not the graphs of functions, so we can not discuss the continuity.

Example 6.2:

Discuss the continuity of the function

$$f(x) = \begin{cases} 2x & \text{for} \quad 0 \leq x < 6 \\ 3x + 1 & \text{for} \quad 6 \leq x < \infty \end{cases}$$

at $x = 6$.

Solution:

Value of the function at $x = 6$:

$$f(6) = 3\,x\,6 + 1 = 19$$

Left hand imit at 6:

$$\lim_{x \to 6^-} f(x) = \lim_{x \to 6^-} (2x) = 12$$

Right hand limit at 6:

$$\lim_{x \to 6^+} f(x) = \lim_{x \to 6^+} (3x + 1) = 19$$

Limit at 6:

$$\lim_{x \to 6^-} f(x) \neq \lim_{x \to 6^+} f(x)$$

$\lim_{x \to 6} f(x)$ does not exist.

So that the function $f(x)$ is discontinuous at $x = 6$.

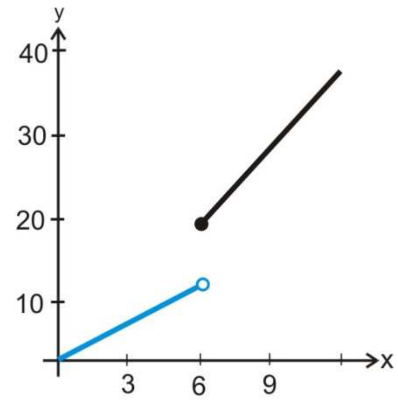


Figure 6.4

Figure 6.4

ONE SIDED CONTINUOUS FUNCTIONS:

A function f is continuous from the right at a number "a" if

$$\lim_{x\to a^+} f(x) = f(a)$$

A function f is continuous from the left at "a" if

$$\lim_{x\to a^-} f(x) = f(a)$$

Examples 6.4:

The Greatest integer function $f(x) = \lfloor x \rfloor$ is continuous from the right at each integer "a" because

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \lfloor x \rfloor = a = f(a)$$

but $f(x)$ is discontinuous from the left at each integer "a" because

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \lfloor x \rfloor = a \neq f(a)$$

Theorem A-3:

If two functions f and g are continuous at "a" then the following functions are also continuous.

(i) f +g (ii) f −g (iii) bf , where b is a constant.

(iv) fg (v) $\frac{f}{g}$ if $g(a) \neq 0$.

Proof:

Since f and g are continuous at "a", so

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a)$$

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a}\{f(x)+g(x)\}$$

$$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$= f(a) + g(a)$$

$$= (f+g)(a)$$

Hence $f + g$ is continuous at "a"

The proof of (ii), (iii),(iv), (v) are left for students.

Examples 6.5:

Show that a polynomial function is continuous for all $x \in R$.

Solution:-

Using the result (i) and (ii) of theorem A-3, we will prove that a polynomial is continuous for all $x \in R$.

Consider a polynomial $P(x)$

$$P(x) = c_{n^{x^n}} + c_{n-1^{x^{n-1}}} + \cdots + c_{2^{x^2}} + c_1x_1 + c_o$$

where $c_o, c_1, c_2, \cdots, c_{n-1}, c_n$ are constants.

We discuss the continuity at $x = a$
Let $f(x) = x^m$

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^m = a^m = f(a)$$

$\Rightarrow f(x) = x^m$ is continuous at $x = a$.
let $g(x) = c_o$

$$\lim_{x\to a} g(x) = \lim_{x\to a} c_o = c_o = g(a)$$

$\therefore\ g(x) = c_o$ is continuous at $x = a$.
$f(x) = x^m$ is continuous function then by the theorem A-3 (iii) $c_m x^m$ is continuous.
Since $P(x)$ is the sum of the function of the form $c_m x^m$ and a constant function it follows from theorem A-3 (i) that $P(x)$ is continuous at $a \in R$.

Example 6.6:
Show that a rational function is continuous wherever it is defined, that is continuous on its domain.
Solution:
Consider a rational function $f(x)$

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$, $Q(x)$ are polynomials. The domain of f is D

$$D = \{x \in R \mid Q(x) \neq 0\}$$

According to above example $P(x)$ and $Q(x)$ are continuous for all $x \in D \subset R$,by theorem A-3(v), f is continuous for all $x \in D$.