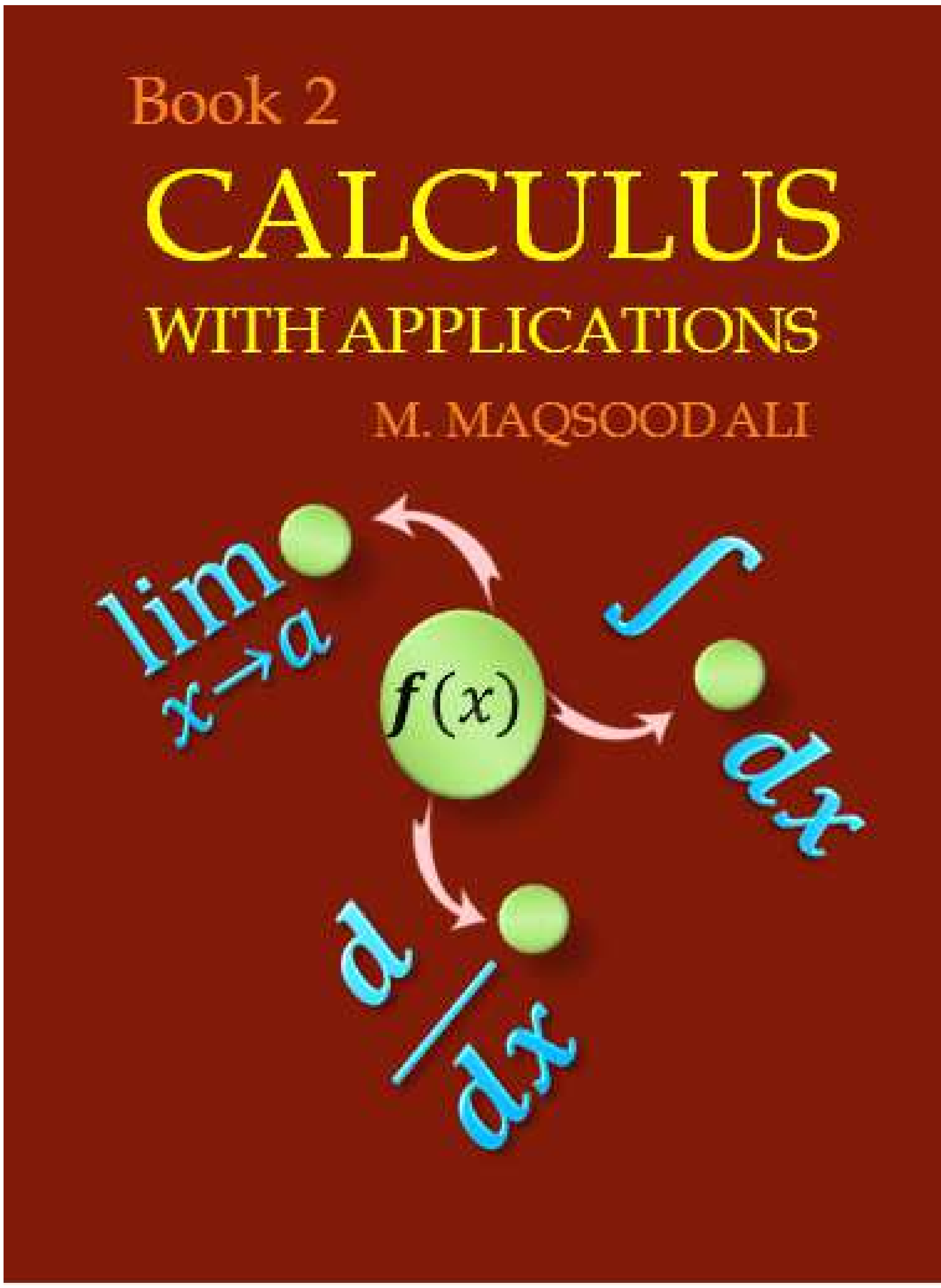
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

Chapter 5

LIMITS

INDETERMINATE FORMS:

The undefined values

$$\frac{0}{0}\ ,\ 0.\infty\ ,\ 0^0\ ,\ \frac{\infty}{\infty}\ ,\ \infty-\infty\ ,\ \infty^\infty\ ,\ 1^\infty$$

of a function $f(x)$ at $x = a$ are called indeterminate forms.

LIMIT OF THE FUNCTION

Definition 1:

A real number "l" is the limit of a function f at "a", if $f(x)$ gets closer and closer to "l" as x approaches "a". It is written as

$$\lim_{x\to a} f(x) = l$$

Explanation:

$$\lim_{x\to a} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = l$$

Figure 5.1

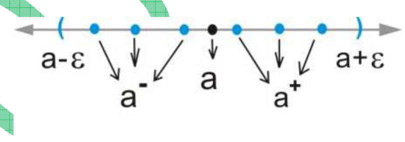

Figure 5.1

where a^- and a^+ lie in the deleted neighborhood of "a" on a real number line in left and right side of "a" respectively. Thus, $a^+ \cong$ a $\cong a^-$.

Definition 2:

A real number "l" is the limit of a function f at "a". If for every real number $\varepsilon > 0$ there exist a coressponding real number $\delta > 0$, such that

$$|x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

Value and Limit of the Functions:

Difference between value of the function and limit of the function can be understand by the following function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

<u>Value of the function at 3:</u>

Substitute $x = 3$

$$f(3) = \frac{3^2 - 9}{3 - 3}$$

$$= \frac{0}{0} \quad \text{(undefined)}$$

<u>Explanation:</u>

Simplify the function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

$$= \frac{(x + 3)(x - 3)}{(x - 3)}$$

$$\neq (x + 3)(1) \quad \text{for} \quad x = 3$$

because for $x = 3$

$$\frac{x - 3}{x - 3} = \frac{3 - 3}{3 - 3}$$

$$= \frac{0}{0} \neq 1$$

<u>Limit of the function at 3:</u>

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2 - 9}{x - 3}$$

$$= \lim_{x\to 3} \frac{(x + 3)(x - 3)}{x - 3}$$

$$= \lim_{x\to 3} (x + 3)(1)$$

$$= 6$$

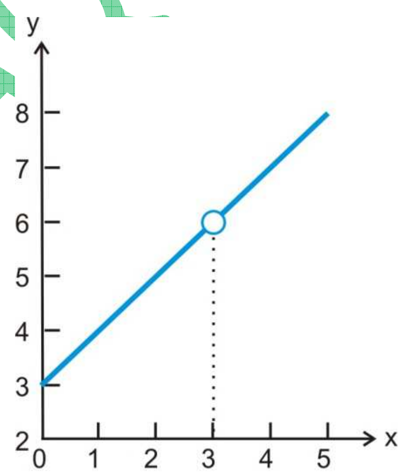

Figure 5.2

<u>Explanation:</u>

$$x \to 3 \quad \Rightarrow \quad \frac{x - 3}{x - 3} = 1 \neq \frac{0}{0}$$

because $x \to 3$ means x is very nearly equal to 3 not exactly equal to 3. So x is either less than 3 $(x < 3)$ or greater than 3 $(x > 3)$.

If $x = 3.00\ldots01 > 3$, then

$$\frac{x - 3}{x - 3} = \frac{3.00\ldots01 - 3}{3.00\ldots01 - 3}$$

$$= \frac{0.00\ldots01}{0.00\ldots01}$$

$$= 1 \neq \frac{0}{0}$$

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LEFT AND RIGHT HAND LIMITS

Left Hand Limit:

Left hand limit of a function f is "l" at "a" and written as

$$\lim_{x \to a^-} f(x) = l$$

a^- means, it is not equal to negative a $(-a)$ but it is just on left side of "a" in the neighborhood of "a". It can be defined as

$$a^- \in (a - \varepsilon, a)$$

or $\quad a - \varepsilon < a^- < a$

where $(a - \varepsilon, a + \varepsilon)$ is the neighbourhood of "a".

Right Hand Limit:

Right hand limit of a function is "l" at "a" written as

$$\lim_{x \to a^+} f(x) = l$$

a^+ means, it is not equal to positive a $(+a)$ but it is just on right side of "a" in the neighborhood of a. It can be defined as

$$a^+ \in (a, a + \varepsilon)$$

or $\quad a < a^+ < a + \varepsilon$

On a number line

figure 5.4.

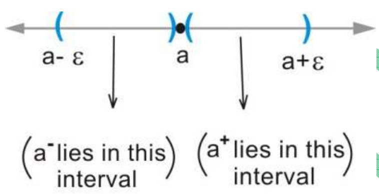

Figure 5.4

Limit of the Function:

Limit of the function at "a" ($\lim_{x \to a} f(x)$) exist only when the left and right hand limits are equal.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

So that, if

$$\lim_{x \to a^-} f(x) = l = \lim_{x \to a^+} f(x)$$

then

$$\lim_{x \to a} f(x) = l$$

Example 5.2 : Discuss the limit of the function

$$f(x) = \begin{cases} 2x + 5 & \text{for } x < 5 \\ 2x & \text{for } x \geq 5 \end{cases}$$

(1) at 5 (2) at 6

Solution:

(1) $\quad \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (2x + 5) = 15$

$\quad \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} 2x = 10$

Left hand limit $\neq$ Right hand limit

So the limit of $f(x)$ does not exist at 5.

Figure 5.5

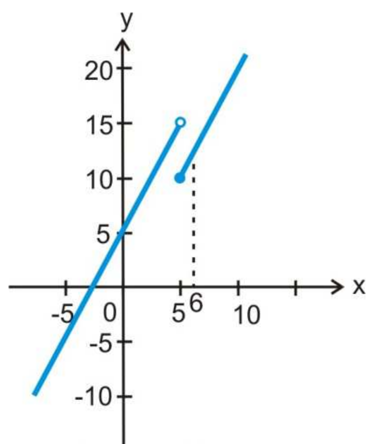

Figure 5.5

(2) $\lim_{x\to 6^-} f(x) = \lim_{x\to 6^-} 2x = 12$

$\lim_{x\to 6^+} f(x) = \lim_{x\to 6^+} 2x = 12$

Left hand limit = Right hand limit = 2

$\lim_{x\to 6} f(x) = 12$

Figure 5.5

Example 5.3:

Discuss the limit of Greatest integer function at 2.

Solution:

$$f(x) = \lfloor x \rfloor$$

Greatest integer function is defined as

$f(x) = \lfloor x \rfloor = a$ for $a \le x < a + 1,\ a \in Z$ and $x \in R$

so that

$f(x) = \lfloor x \rfloor = 0$ for $0 \le x < 1$ or $x \in [0,\ 1)$

$= 1$ for $1 \le x < 2$ or $x \in [1,\ 2)$

$= 2$ for $2 \le x < 3$ or $x \in [2,\ 3)$

$= 3$ for $3 \le x < 4$ or $x \in [3,\ 4)$

Left hand limit at 2:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \lfloor x \rfloor = 1$$

{because $2^- > 1$ and $2^- < 2 \Rightarrow 2^- \in [1,\ 2) \Rightarrow f(2^-) = 1$}

Right hand limit at 2:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \lfloor x \rfloor = 2$$

{because $2^+ > 2$ but $2^+ < 3 \Rightarrow 2^+ \in [2,\ 3) \Rightarrow f(2^+) = 2$}

Limit at 2:

$$\lim_{x\to 2^-} f(x) \ne \lim_{x\to 2^+} \lfloor x \rfloor$$

Figure 5.6

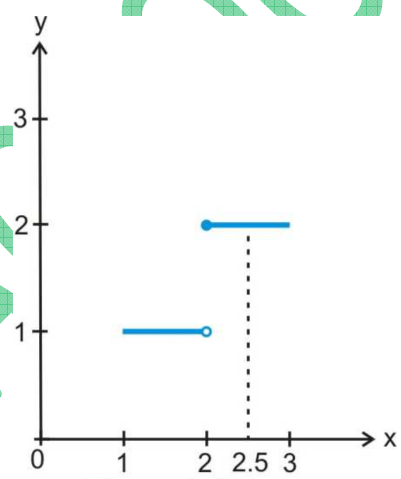

Figure 5.6

Hence $\lim_{x\to 2} f(x)$ does not exist.

Example 5.4:

Discuss the Limit of Greatest integer function at 2.5.

Solution: **Left hand limit at 2.5:**

$$\lim_{x\to 2.5^-} f(x) = \lim_{x\to 2.5^-} \lfloor x \rfloor = 2$$

{because $2.5^- < 2.5$ but $2.5^- > 2 \Rightarrow 2.5^- \in [2,\ 3) \Rightarrow f(2.5^-) = 2]$}

Right hand limit at 2.5:

$$\lim_{x\to 2.5^+} f(x) = \lim_{x\to 2.5^+} \lfloor x \rfloor = 2$$

{since $2.5^+ > 2.5$ but $2.5^+ < 3 \Rightarrow 2.5^+ \in [2,\ 3)$ $\Rightarrow f(2.5^+) = 2$}

Limit at 2.5:

$$\lim_{x\to 2.5^-} f(x) = \lim_{x\to 2.5^+} f(x) = 2$$

Figure 5.6

Hence

$$\lim_{x\to 2.5} f(x) = 2$$

Example 5.5:

Find the limit of modulus function at zero, **figure 5.7**.

Solution:

Modulus function is defined as

$$f(x) = |x| = \begin{cases} -x & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ +x & \text{for} \quad x > 0 \end{cases}$$

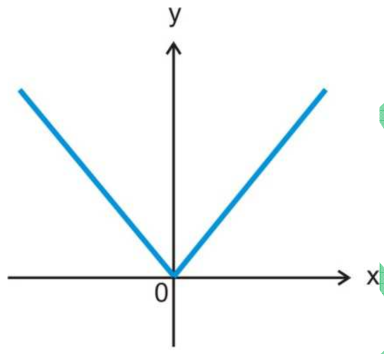

Figure 5.7

Left hand limit at zero:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-}(-x)$$

{because $0^- < 0 \Rightarrow |x| = -x$}

Right hand limit at zero:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+}(+x) = 0$$

{because $0^+ > 0 \Rightarrow |x| = +x$}

Limit at zero

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$$

Hence

$$\lim_{x\to 0} f(x) = 0$$

Example 5.6:

Find the limit of modulus function at 2, **figure 5.8**.

Solution:

$$f(x) = |x|$$

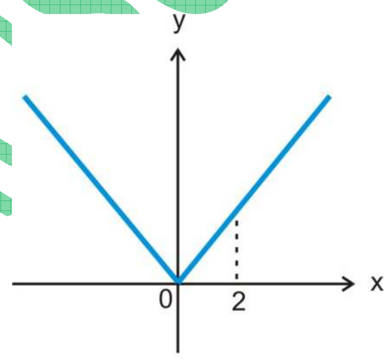

Figure 5.8

Left hand limit at 2:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(+x) = 2$$

{because $2^- > 0 \Rightarrow |x| = +x$}

Right hand limit at 2:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(+x) = 2$$

{because $2^+ > 0 \Rightarrow |x| = +x$}

Limit at 2:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 2$$

Hence

$$\lim_{x\to 2} f(x) = 2$$

DE L' HOPITAL RULE

Theorem:

$f(x)$ and $g(x)$ are two functions if the derivatives of both functions exists and $f(a) = 0 = g(a)$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Proof:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{f(a+h)}{g(a+h)} \quad \to (1)$$

By Lagrange's mean value theorem

$$f(a+h) = f(a) + h\,f'(a+\theta h),\ 0 < \theta < 1$$

Hence equation (1) can be written as

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\left(f(a) + hf'(a+\theta_1 h)\right)}{\left(g(a) + hg'(a+\theta_2 h)\right)}$$

where $0 < \theta_1 < 1$ *and* $0 < \theta_2 < 1$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{hf'(a+\theta_1 h)}{hg'(a+\theta_2 h)}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

In general if the nth derivative of the function exists and

$$f(a) = f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0$$

$$g(a) = g'(a) = g''(a) = \cdots = g^{(n-1)}(a) = 0$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^n(x)}{g^n(x)}$$

Example 5.7 :

Evaluate the following limit

$$\lim_{x\to\pi/4}\frac{sec^2x - 2tanx}{sin4x}$$

Figure 5.9

Solution:

$$\lim_{x\to\pi/4}\frac{sec^2x2tanx}{sin4x},\quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\pi/4}\frac{2sec^2xtanx - 2sec^2x}{4cos4x}$$

$$= \frac{0}{-4} = 0$$

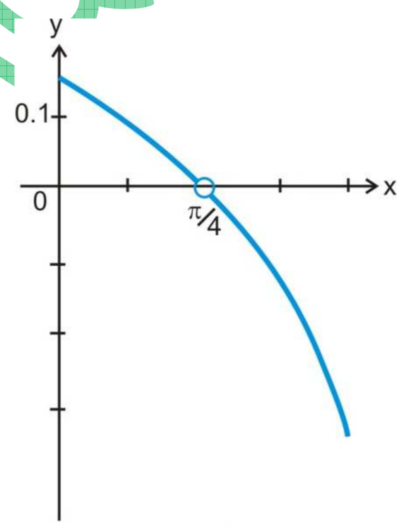

Figure 5.9

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$= \lim_{x\to\pi/2}[\frac{1}{2cosx}]$

$= 0$

$$y = e^0 = 1$$

EXERCISE 5

De l ' Hopital rule is applicable if

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$$

Show that this rule is applicable for the following.

(1) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$,

(2) $\lim_{x\to a}[f(x)\,.\,g(x)] = 0\ \times\infty$

(3) $\lim_{x\to a}[f(x) - g(x) = \infty - \infty$

(4) $\lim_{x\to\infty}[f\big(x)^{g(x)}\big] = 0^0$

(5) $\lim_{x\to a}[f(x)]^{g(x)} = 1^\infty$

(6) $\lim_{x\to a}[f(x)g^{(x)} = \infty^0$

Evaluate:

(7) $\lim_{x\to 0}(\frac{1}{x^2} - cosec^2x)$

(8) $\lim_{x\to 2}(2-\frac{x}{2})^{\tan(\frac{\pi x}{4})}$

(9) $\lim_{x\to 0}\left[\frac{a}{(e^{2ax}-1)x} - \frac{1-ax}{2x^2}\right]$

(10) $\lim_{x\to 0}\frac{ax^2-b^{x^2}}{x^2}$

(11) $\lim_{x\to 0}\left(\frac{1}{x}cotx\right)^{\frac{1}{x}}$

(12) $\lim_{x\to 0}(coshx)^{cothx}$