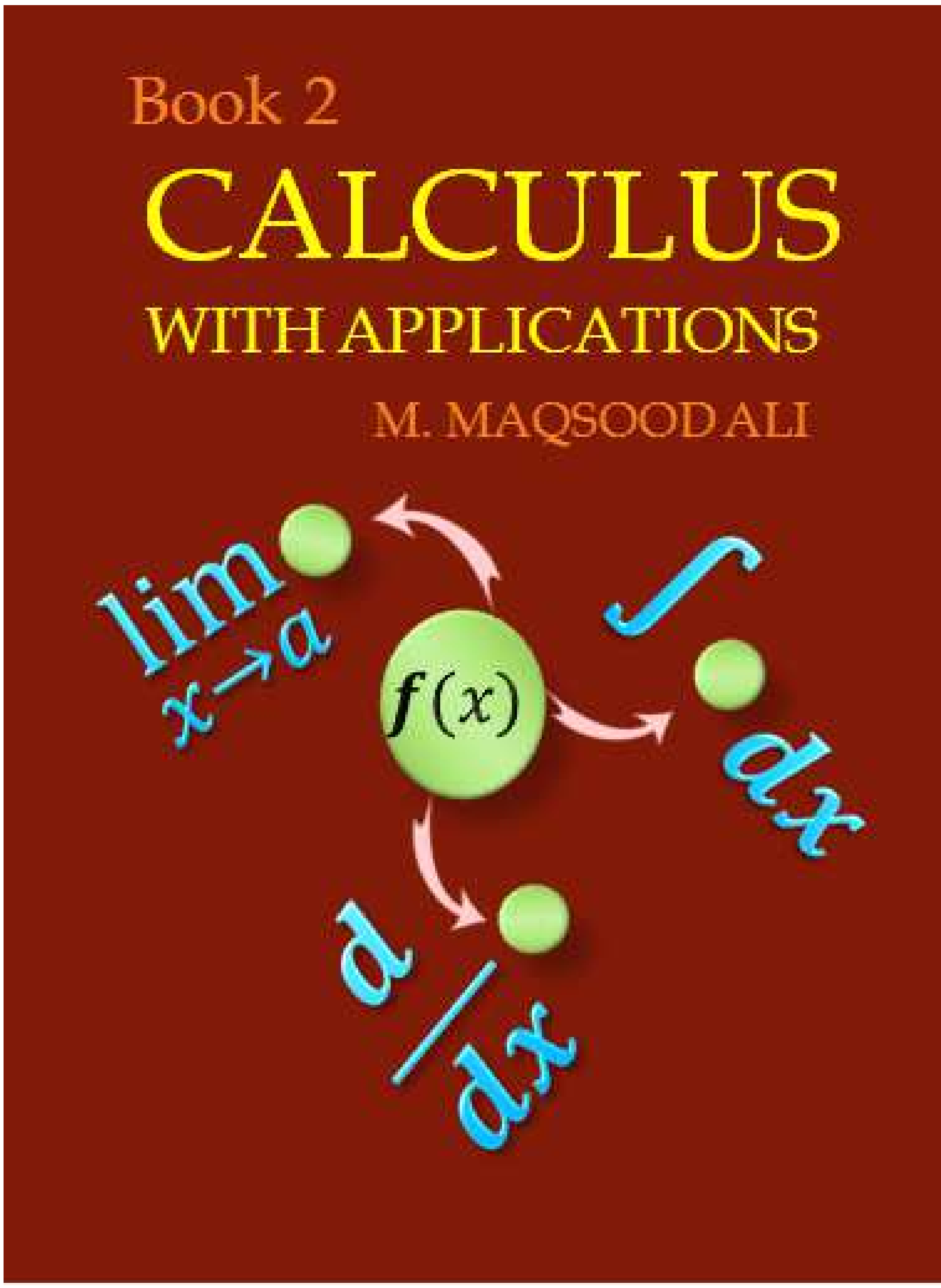
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

Chapter 4

FUNCTIONS

Machine is acting a very important role in our life. The work of machine is that we input any object into the machine. The machine proceeds this object according to the nature of the machine. The output which we get has a change according to the type of machine. For example,
The washing machine washes the cloth.
The sewing machine sews the cloth.
The knitting machine knits different design on the machine.

In above examples, the input are same that is cloth but the output are different. The output depends on the type of the machine.

A function is a fictitious machine that produces a unique output number for a given input number. If x is the input and f is the function then output we obtain is denoted by $f(x)$. Following are the examples of functions.

$f(x) = 5x + 3, \forall x \in \mathbb{R}$, is a straight line.
$f(x) = 2x^2 + 3x + 1, \forall x \in \mathbb{R}$, is a parabola.
$f(x) = x^3, \forall x \in \mathbb{R}$, is a curve.

To understand the definition of the function.
Firstly we discuss about relation and mapping.

Relation

A relation is a set of ordered pairs.

OR

Any subset of the set of Cartesian product A X B is called a relation.

Example: If A = {1, 3, 5} and B = {2, 4, 6} Then the following subsets are relation between A and B.

(1) $\{(1,2),(1,4),(2,6)\}$
(2) $\{(3,4),(1,6),(3,4),(5,6)\}$
(4) $\{(1,2),(3,4),(5,6)\}$
(5) $\{(1,6),(3,4),(5,2),(5,4),(5,6)\}$

In above examples we see that an element of A may be related with more than one element of B.

The points can be plotted on a graph paper.

MAPPING

Mapping illustrates a relation geometrically by marking points on two sets A and B and drawing arrows from the elements of set A to the elements of set B, as shown in the figures.

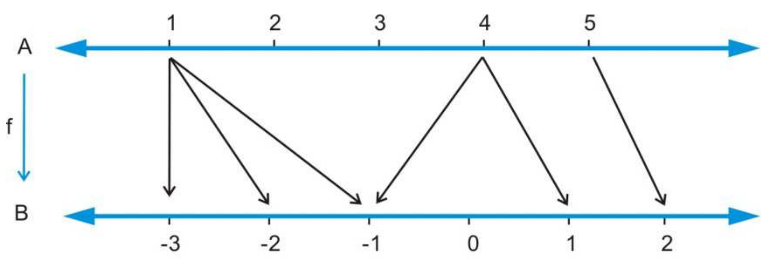

Figure 4.1

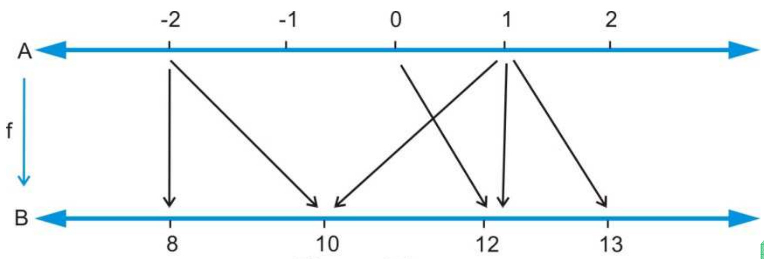

Figure 4.2

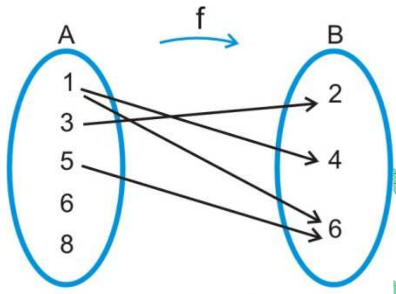

Figure 4.3

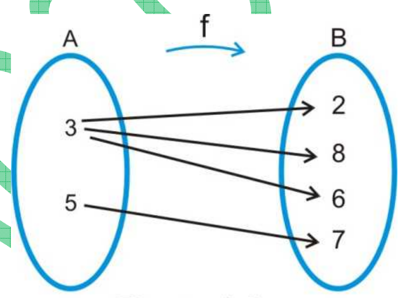

Figure 4.4

In the **figures 4.1, 4.2, 4.3, 4.4 and 4.5**, set A is being mapped into set B. It illuminates

(I) Two or more arrows may begin at the same element of A, may end at the different elements of B.

(II) The arrows begin at the different elements of A, may end at the same element of B.

(III) It is not necessary that all elements of A should map into the set B.

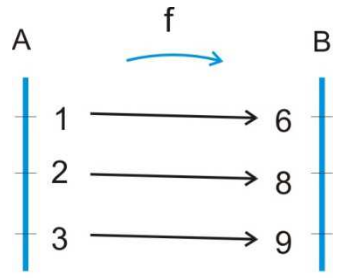

Figure 4.5

FUNCTIONS

Definition 1:

A function of assign to each element x of set A an element $f(x)$ of another set B.

$$f: A \to B \ , \ \forall x \in A \ , \ f(x) \in B$$

Examples 4.1:

(1) f is a function A into B.

Figure 4.6

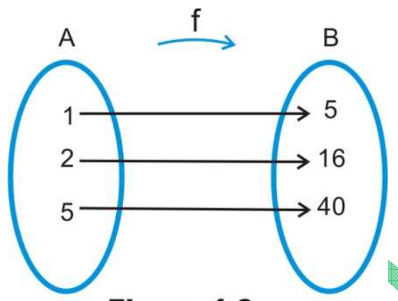

Figure 4.6

(2) f is a function A into B.

Figure 4.7

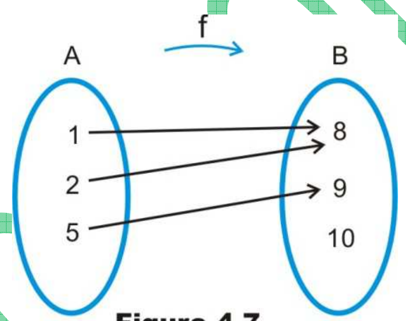

Figure 4.7

(3) f is not a function A into B.

Figure 4.8

f is not a function A into B, because f does not assign to 5 of set A an element of set B. It is a mapping only.

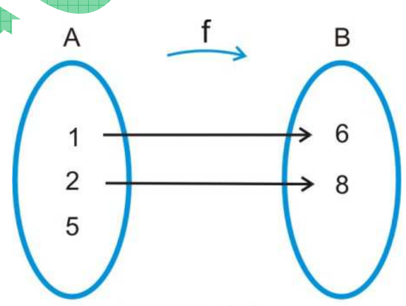

Figure 4.8

(4) f is a function A into B.

Figure 4.9

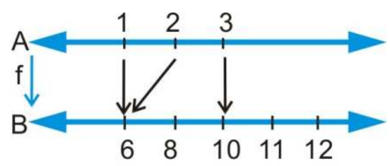

Figure 4.9

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DOMAIN AND RANGE

A function, A into B can be written as

$$f : x \mapsto y$$

where $x \in A$ and $y \in B$. The value of the function at $x \in A$ is denoted as $f(x) = y$

Set A is called domain of f and $y \in B$ is the image of x under the function. The set of corresponding values $y \in B$ of $x \in A$ is called range of the function.

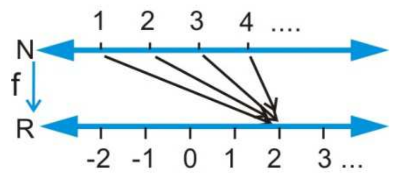

Figure 4.15

Example 4.3:

If $A = \{1,2,3\}$ *and* $B = \{1,2,3,4,5,6,7,8,9,10\}$
f is a function A into B defined as

$$f : x \mapsto x^2$$

find the domain and range of f.

Solution:

$$f(x) = x^2$$

$\Rightarrow$ $f(1) = 1$, $f(2) = 4$, $f(3) = 9$

Domain of $f = \{1,2,3\}$

Range of $f = \{1,4,9\}$

Example 4.4:

A function $f : \mathbb{N} \to \mathbb{R}$ is defined as

$$f : x \mapsto 2$$

find the domain and range of f, **figure 4.15.**

Solution:

Since $f(x) = 2$

Domain of $f = \mathbb{N}$

Range of $f = \{2\}$

DOMAIN OF A RADICAL AND RATIONAL FUNCTION

We find the domain of the following functions.

(1) f(x) = $\sqrt{x-1}$, f is real valued function.

$\sqrt{x-1}$ is a real number , if

$x - 1 \geq 0$

$\Rightarrow$ $x \geq 1$

Domain of f is $x \geq 1$, x is a real number.

(2) $f(x) = \sqrt{x^2-1}$, f is a real valued function.

$\sqrt{x^2-1}$ is a real number, if

$x^2 - 1 \geq 0$

$\Rightarrow$ $x^2 \geq 1$

$\Rightarrow$ $x \leq -1$ and $x \geq 1$, is the domain of f.

(1) $f(x) = \dfrac{x^2 + 3}{x - 1}$

$f(x)$ exist if $x - 1 \neq 0 \Rightarrow x \neq 1$

Domain of $f = \mathbb{R} - \{1\}$

(2) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 1}$

$f(x)$ exist if $x^2 - 1 \neq 0 \Rightarrow x^2 \neq 1 \Rightarrow x \neq \pm 1$

$\therefore$ domain of $f = \mathbb{R} - \{-1, 1\}$

(3) $f(x) = \dfrac{x^2 - 3}{x^2 + 1}$

$f(x)$ exist if $x^2 + 1 \neq 0 \Rightarrow x^2 \neq -1 \Rightarrow x \neq \pm\sqrt{-1} \notin \mathbb{R}$

$\therefore$ domain of $f = \mathbb{R}$.

ONE TO ONE FUNCTIONS

A function $f : A \rightarrow B$ is said to be one to one if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$ such that $x_1, x_2 \in$ A and $f(x_1), f(x_2) \in$ B.

Examples 4.5:

(a) Functions from **(1) to (6)** are one to one functions.

(1)

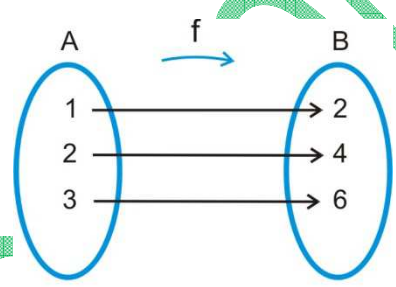

Figure 4.16

(2)

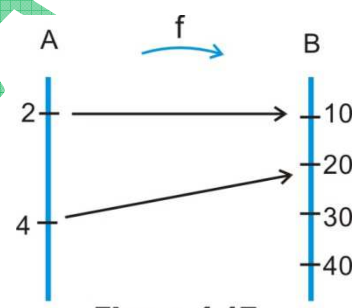

Figure 4.17

(3)

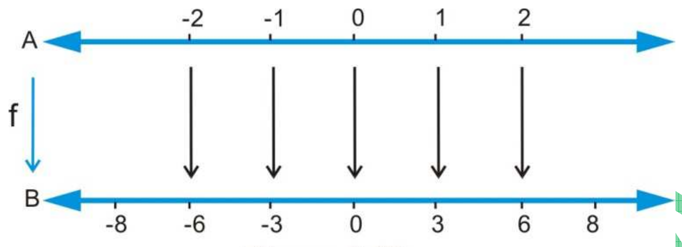

Figure 4.18

(4) If A = {1, 2, 4} and B = {1, 10, 20}

and $f = \{(1, 1), (2, 10), (4, 20)\}$

(5) $f(x) = 5x, \forall\, x \in \mathbb{R}$

(6) $f(x) = x^2 + 3x, \forall\, x \in \mathbb{R}$

(b) Following functions are not one-one functions.

(1) $f(x) = |x|$

(2) $f(x) = 6$ (or any constant function)

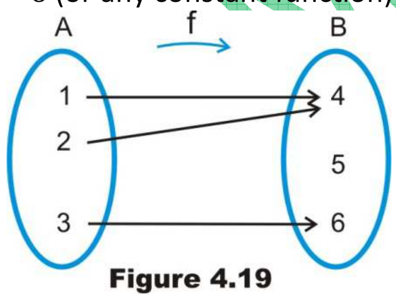

Figure 4.19

ONTO FUNCTIONS

A function $f: A \mapsto B$ is said to be onto if for every $y \in B$ there is some x ∈ B such that $y = f(x)$.

Example 4.6:

(a) Functions (1) and (2) are onto functions.

(1) **Figure 4.20**

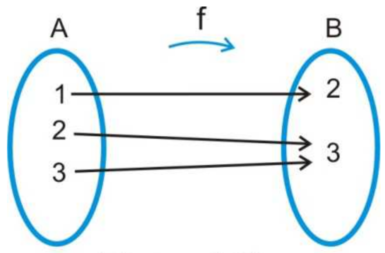

Figure 4.20

Because each element of B is the image of some element of A, under f.

(2)

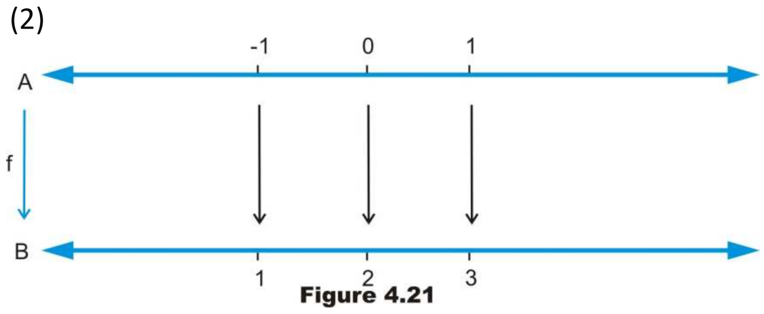

Figure 4.21

Because each element of B is the image of an element of A, under f.

(b) Following functions are not onto functions.

(1) **Figure 4.22**

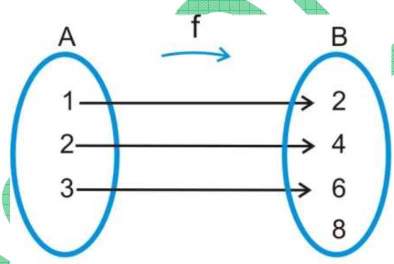

Figure 4.22

Because 8 ∈ B is not the image of any element of A under f.

(2)

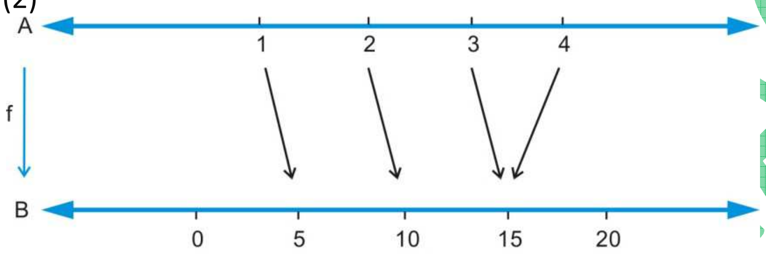

Figure 4.23

Because 0 and 20 are not the image of any element of A under f.

(3) If A = {2, 4, 6, 8} and B = {0, 10, 20, 30, 40, 50, 60} are two sets and f is a function A into B is defined by

$$f: x \mapsto 5x \quad , \quad \forall x \in A$$

Since $\quad f(x) = 5x \ , \ x \in A$

$f(2) = 10, f(4) = 20, f(6) = 30$ *and* $f(8) = 40$

As 0 and 60 are not the image of any element of A, so f is not a onto function.

EVEN FUNCTIONS

A function f is said to be even function, if

$$f(-x) = f(x).$$

Example 4.7:

$f(x) = 3x^4 + 2x^2$ is an even function, because

$$\begin{aligned} f(-x) &= 3(-x)^4 + 2(-x)^2 \\ &= 3x^4 + 2x^2 \\ &= f(x) \end{aligned}$$

Figure 4.24

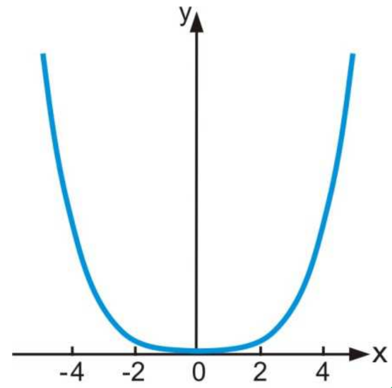

Figure 4.24

ODD FUNCTIONS

A function f is said to be odd function, if

$$f\,(-x) = -f(x).$$

Example 4.8: $f(x) = 3x^3 + 2x$ is an odd function, because

$$\begin{aligned} f(-x) &= 3(-x)^3 + 2(-x) \\ &= -3x^3 - 2x \\ &= -f(x) \end{aligned}$$

Figure 4.25

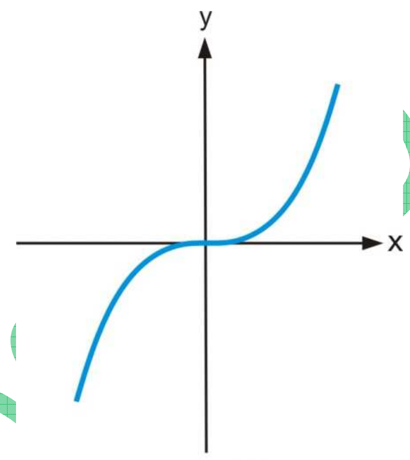

Figure 4.25

COMBNATIONS OF FUNCTIONS

Functions may be combined algebraically to form a new function. Sum Difference, Product and Quotient of two functions f and g are defined as

(1) Sum: $(f + g)(x) = f(x) + g(x)$

(2) Difference: $(f - g)(x) = f(x) - g(x)$

(3) Product: $(f\,.\,g)(x) = f(x)\,.\,g(x)$

(4) $Quotient$: $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

Example 4.9: If $f(x) = x^3$ and $g(x) = x + 1$, find

(i) $(f + g)(x)$ (ii) $(f - g)(x)$

(iii) $(fg)(x)$ (iv) $(\frac{f}{g})(x)$

Solution:

(i) $(f + g)(x) = f(x) + g(x)$
$= x^3 + x + 1$

(ii) $(f - g)(x) = f(x) - g(x)$
$= x^3 - x - 1$

(iii) $(f\,.\,g)(x) = f(x)\,.\,g(x)$
$= x^3\,(x + 1)$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$= (x^2 + 2) + 1$, *substituting* $g(x) = x^2 + 2$

$= x^2 + 3$

Example 4.11:

Find $(h_0 g_0 f)(x)$, if

$f(x) = x^2, \quad g(x) = x + 1, \quad h(x) = \sqrt{x}$

Solution:

$(h_0 g_0 f)(x) = h[(g_0 f)(x)]$

$= h[g(f(x))]$

$= h[f(x) + 1]$, substituting $x = f(x)$ into $g(x)$

$= h(x^2 + 1)$, *substituting* $f(x) = x^2$

$= \sqrt{x^2 + 1}$, substituting $x = x^2 + 1$ *into* $h(x)$

Note:

(1) $f_0 g \neq g_0 f$ {see example 1}

So functions composition is not commutative.

(2) $f_0(g_0 h) = (f_0 g)_0 h$

So functions composition is associative.

Example 4.12:

Express $h(x) = \sqrt{x^2 + 3}$ as a composition of two functions.

Solution:

Method 1:

To evaluate h(x), first the value of $x^2 + 3$. So let

$$f(x) = x^2 + 3$$

Now, we compute square root of this value. So

$$g(x) = \sqrt{x} \text{ or } g(f(x)) = \sqrt{f(x)}$$

The composite function is

$$h(x) = \sqrt{f(x)} = g[f(x)] = (g_0 f)(x)$$

Method 2:

Let $u = f(x) = x^2 + 3$

$\Rightarrow g(u) = \sqrt{u}$

So, that

$$h(x) = \sqrt{x^2 + 3} = \sqrt{u} = g(u) = g[f(x)]$$
$$= (g_0 f)(x)$$

GRAPHS OF FUNCTIONS

You are watching a cricket match on the television. You will have been observed that a graph is shown on the T.V. during the match as shown below.

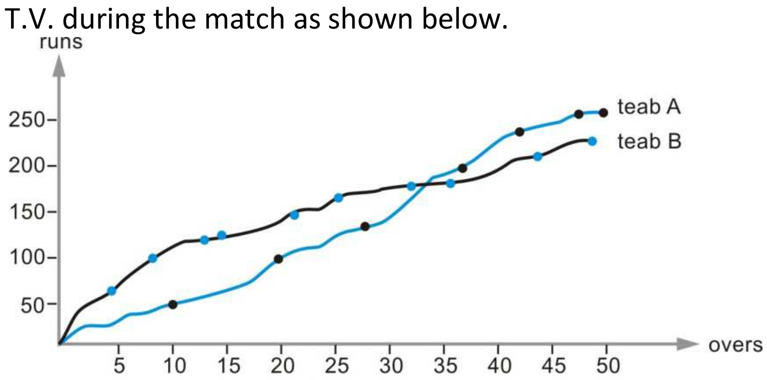

Figure 4.29

By putting a glance on the graph you understand all the situation of the match that which team played well in particular intervals.

The graph is a way to understand the nature of the function or data in a glance.

VERTICAL LINE TEST

A vertical line cannot intersect the graph of the function more than once.

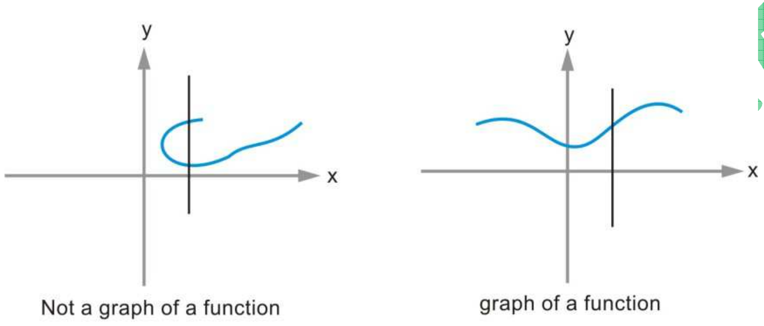

Figure 4.30