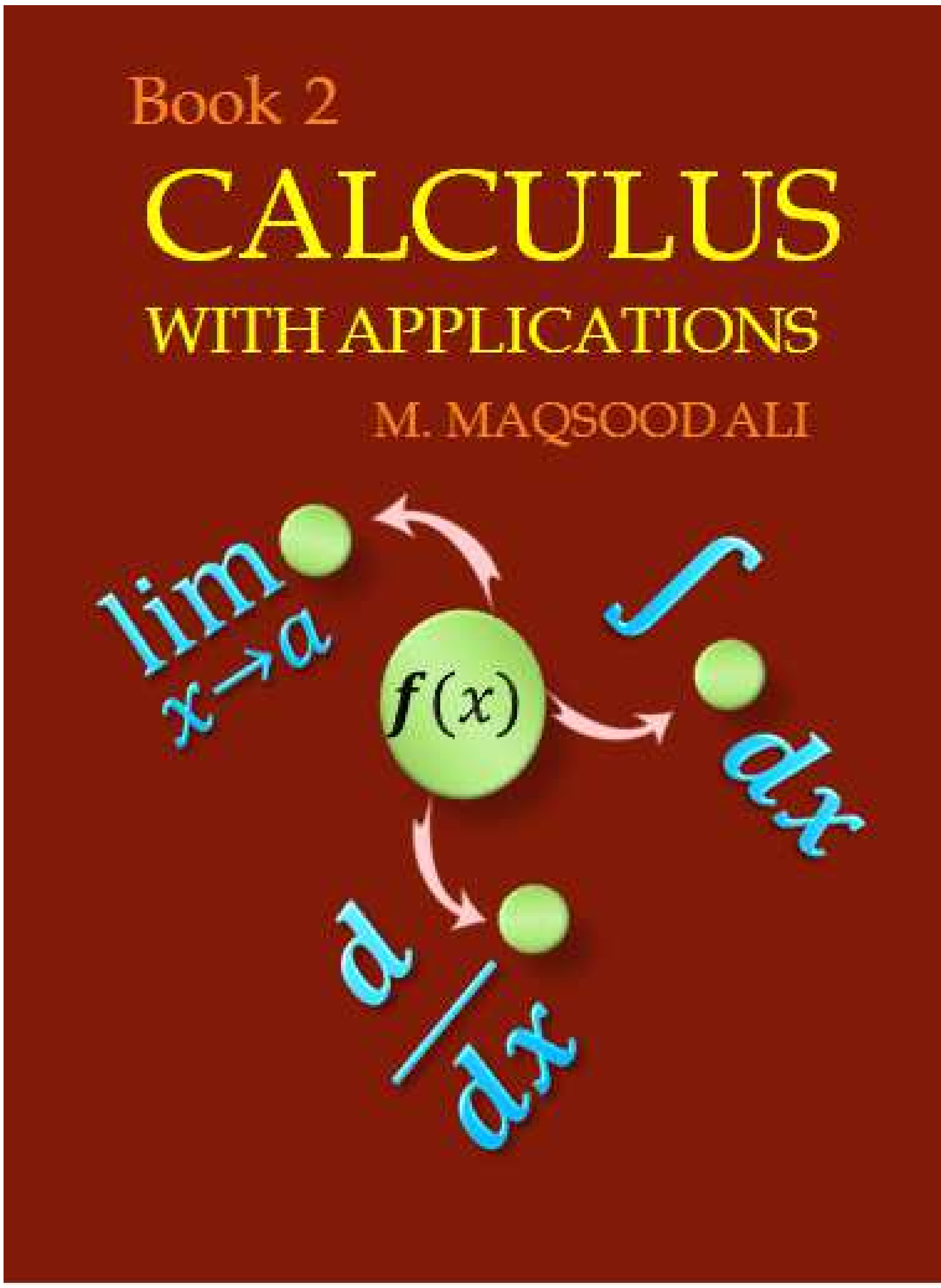
Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

Chapter 3

BOUNDS

SETS AND NUBER LINE:

A **number line** is a graphical representation of **real numbers**. All real numbers lie on this line.

Figure 3.1.

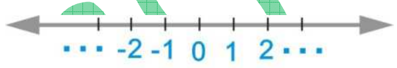

Figure 3.1

Set of natural numbers $\mathbb{N}$, set of whole numbers $\mathbb{W}$, set of integers $\mathbb{Z}$, set of rational numbers $\mathbb{Q}$ and set of irrational numbers $\mathbb{Q}'$ are the subsets of the set of real numbers $\mathbb{R}$. All the numbers belongs to set of integers $\mathbb{Z}$ can be represented on the number line at there exact position. An integer on the number line is represented by a solid circle ($\bullet$). The integers $-4, 0, 2, \text{and } 5$ are represented on a number line as

figure 3.2.

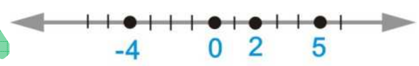

Figure 3.2

All the **subsets** of the **set of integers** can also be represented very easily by a **number line**.

(i) **Set of natural numbers:**

$$\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}$$

Natural numbers are represented on a number line as

figure 3.3.

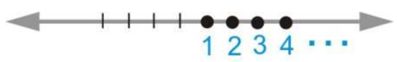

Figure 3.3

(ii) **Set of whole numbers:**

$$\mathbb{W} = \{0, 1, 2, 3, 4, \cdots\}$$

Whole numbers are represented on a number line as

figure3.4.

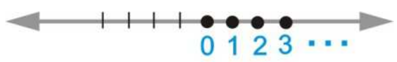

Figure 3.4

(iii) **Set of negative integers:**

$$Z^- = \{-1, -2, -3, -4, \cdots\}$$

Negative integers are represented on a number line as

figure 3.5

Figure 3.5

(iv) **Set of integers:**

$$\mathbb{Z} = \{-\infty, \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots, +\infty\}$$

Integers are represented on a number line as

figure 3.6

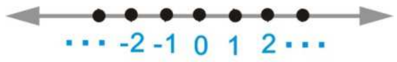

Figure 3.6

A lot of **rational numbers** and all **irrational numbers** cannot express at their exact position on the **number line**. The sets of rational and irrational numbers are written as

set of rational numbers

$$\mathbb{Q} = \left\{x \mid x = \frac{p}{q}\ ,\ \ p, q \in Z,\ \ q \neq 0\right\}$$

set of irrational numbers

$$\mathbb{Q}' = \left\{x \mid x \neq \frac{p}{q}\ ,\ \ p, q \in Z,\ \ q \neq 0\right\}$$

The rational numbers and irrational numbers completely both togather fill the number line which represents all real numbers. The **set of real numbers** is the union of rational and irrational numbers

$$\mathbb{R} = \{x \mid x \in \mathbb{Q} \cup \mathbb{Q}'\}$$

INEQUALITIES

A number p on the number line must be less than the number q, if p lies left to the number q on the number line. It is written as p < q. We can also say it that q is greater than p and it is written as q > p. The mathematical representation for this is

figure 3.7.

Figure 3.7

The inequality signs are represented by the symbols $<, >, \leq,\ \geq$.

Statements of the form p < q, p ≤ q, q > p and q ≥ p are read as p is less than q, p is less than or equal to q, q is greater than p and q is greater than and equal to p respectively.

The inequalities p < q and p > q are called strict, which do not allow for the possibility of equality and the inequalities p ≤ q and q ≥ p, which allow for the possibility of equality are called non-strict.

A compound or combined inequality p < r < q means p < r and r < q.

INTERVALS

An interval is a subset of the set of real numbers, defined as all the real numbers between two given real numbers.

We will discuss about open interval, closed interval and half open intervals.

Open Interval:

An open interval of two numbers a and b is written as (a, b) where $a < b$. The end points a and b are not included. It can be written as

$$(a, b) = \{x \mid a < x < b\}$$

The graphical representation is **figure 3.8.**

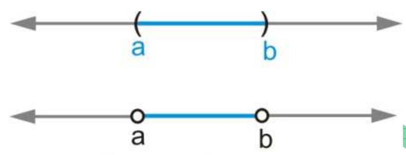

Figure 3.8

Circle ○ and the brackets (,) indicates that the end points are not included in the interval.

Examples 3.1:

(1) $(1, 5) = \{x \mid 1 < x < 5\}$

The graphical representation is **figure 3.9**.

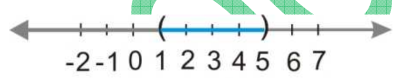

Figure 3.9

(2) $(-2, 3) = \{x \mid -2 < x < 3\}$

The graphical representation is **figure 3.10**.

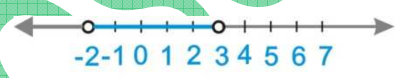

Figure 3.10

Close Interval:

A close interval of two numbers a and b where $a < b$, is written as $[a, b]$. The points a and b are included

$$[a, b] = \{x \mid a \leq x \leq b\}$$

The graphical representation is **figure 3.11.**

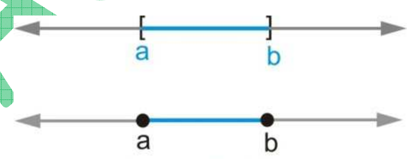

Figure 3.11

Solid circle ● and the brackets [,] indicates that the end points are included in the interval.

Examples 3.2:

(1) $[1, 5] = \{x \mid 1 \leq x \leq 5\}$

The graphical representation is

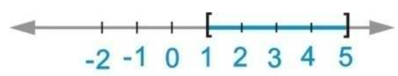

Figure 3.12

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BOUNDS

A set B which is the subset of real numbers $(B \subset R)$. Two given numbers belong to R are called bounds of B and the set B is called bounded set if all the numbers belong to B lies between or equal to the given numbers.

Suppose that $a, b \in R$ and $x \in B$.

The set B is bounded and a, b are bounds of B, such that

$$a \leq x \leq b \quad \text{or} \quad a < x < b\,.$$

UPPER BOUNDS AND BOUNDED ABOVE SET:

A set $B \subset R$ is called bounded above if every number belong to set B is less than or equal to $b \in R$. The number b and the numbers greater than b are said to be upper bounds of B.

Example 3.5:

Consider the sets A, B subsets of R.

(i) $A = \{x \mid -\infty < x < 6\}$

Figure 3.21

Figure 3.21

Since all the numbers in A less than $6 \in R$, so

upper bounds of $A = b \geq 6$

(ii) $B = x \mid -\infty < x \leq 6\}$

Figure 3.22

Figure 3.22

Since all the numbers in B is less than or equal to $6 \in R$, so

upper bounds of $B = b \geq 6$

LOWER BOUNDS AND BOUNDED BELOW SET:

A set $B \subset R$ is called bounded below if every number belongs to B greater than or equal to a number $a \in R$. The number $a \in R$ and all other numbers less than a are said to be lower bounds of B.

Examples 3.6:

Consider the sets A, B subsets of R.

(i) $A = \{x \mid 6 < x < \infty\}$

Figure 3.23

Figure 3.23

Since all the numbers in A greater than $6 \in R$, so

lower bounds of $A = a \leq 6$

(ii) $B = \{x \mid 6 \leq x < \infty\}$

Figure 3.24

Figure 3.24

Since all the numbers in B is greater than or equal to $6 \in R$, so

lower bounds of $B = a \leq 6$

LEAST UPPER BOUND AND GREATEST LOWER BOUND:

If $S \subset \mathbb{R}$, then a number$M \in \mathbb{R}$is said to least upper bound of S,if M is the smallest number in all upper bounds of B.

On subtracting a very small positive value $\varepsilon > 0$ from M there exists at least one number, belong to S ,greater than $M - \varepsilon$.

$M - \varepsilon <$ at least one number of S.

If $S \subset$,$\mathbb{R}$then a number $m \in \mathbb{R}$ is said to be greatest lower bound (glb) of S, if m is the greatest number in all lower bounds of S.

On adding a very small positive number $\varepsilon > 0$ in m there exist at least one number, belong to S, less than $m + \varepsilon$.

$m + \varepsilon >$ at least one number of S.

Example 3.7:

(i) Consider the following set $S \subset R$

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \quad . \; . \quad \; .\right\} \; , \; 0 \notin S$$

Greatest lower bound of $S = m = 0$.

(ii) $S = \{\, x \mid 5 \le x \le 20 \,\}$, $S \subset R$

Figure 3.25

Figure 3.25

Least upper bound of $S = M = 20$

Greater lower bound of $S = m = 5$

(iii) $S = \{\, x \mid 5 < x < 20 \,\}$, $S \subset R$

Figure 3.26

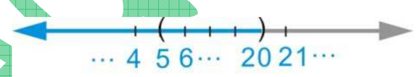

Figure 3.26

Least upper bound of $S = M = 20$

Greater lower bound of $S = m = 5$

Explanation:

Consider a set $A \subset R$

$A = \{\, x \mid 7 \le x \le 15 \,\}$

An **upper bound** is a number greater than or equal to the greatest number of set A.

Upper bounds of $A = b \ge 15$

Least upper bound is the smallest number in upper bounds.

Least upper bound (lub) of $A = M = 15$

A **lower bound** is a number smaller than or equal to the smallest number of set A.

Lower bounds of $A = a \le 7$

Greatest lower bound is the greatest number in lower bounds.

Greatest lower bound of $A = m = 7$

Example 3.8:

Find the upper bounds, lower bounds, least upper bound and greatest lower bound of the following set.

$$A = \{x \mid -5 \leq x \leq 18, x \in R\} \quad , \qquad A \subset R$$

Figure 3.27

Figure 3.27

Upper bounds of $A = b \geq 18$

Lower bounds of $A = a \leq -5$

Least upper bound of $A = M = 18$

Greatest lower bound of $A = m = -5$

BOUNDED SET:

A set is called bounded if it is bounded above and bounded below.

Examples 3.9:

The following sets are bounded.

$A = \{x \mid 6 < x < 19\}$

$B = \{x \mid 6 \leq x \leq 19\}$

$C = \{x \mid 6 < x \leq 19\}$

$D = \{x \mid 6 \leq x < 19\}$

Figure 3.28

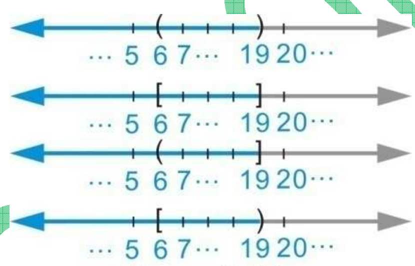

Figure 3.28

The following sets X, Y, Z are not bounded.

$X = \{x \mid -\infty < x < \infty\}$

X is neither bounded above nor bounded below.

$Y = \{x \mid -\infty < x \leq 6\}$

Figure 3.29

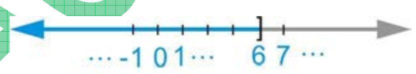

Figure 3.29

Y is bounded above but not bounded below.

$Z = \{x \mid 6 < x < \infty\}$

Figure 3.30

Figure 3.30

Z is bounded below but not bounded above.

NEIGHBOURHOOD

If $a \in \mathbb{R}$ then neighbourhood of a is an open interval with a as its mid-point.

If for $0 < \delta \in \mathbb{R}^+$, then open interval $(a - \delta, a + \delta)$ is a neighbourhood of a. It is the open set

$$\{x \in \mathbb{R} \mid d = \mid x - a \mid < \delta\}$$
$$= \{x \in \mathbb{R} \mid a - \delta < x < a + \delta\}$$

which consists of all points between $a - \delta$ and $a + \delta$.

Figure 3.31a

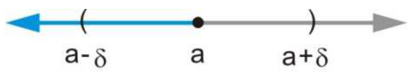

Figure 3.31 a

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MONOTONIC INCREASING SEQUENCE:

A sequence $\{S_n\}$ is called monotonic increasing sequence if

$$S_n \leq S_{n+1} \text{ for } n = 1, 2, 3, \ldots$$

Examples 3.11:

Following sequences are monotonic increasing.

(i) $S_n = \{2, 4, 6, 8, 10, \ldots\}$

Figure 3.37.

(ii) $S_n = \{5, 10, 15, 20, \ldots\}$

(iii) $S_n = \{1, 3, 5, 7, 9, \ldots\}$

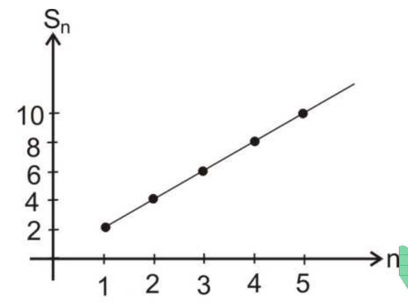

Figure 3.37

MONOTONIC DECREASING SEQUENCE:

A sequence $\{S_n\}$ is called monotonic decreasing sequence if

$$S_n \geq S_{n+1} \text{ for } n = 1, 2, 3, \ldots$$

Examples 3.12:

Following sequences are monotonic decreasing.

$(i)\ \ S_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\right\}$

Figure 3.38

$(ii)\ S_n = \left\{\frac{5}{3}, \frac{5}{6}, \frac{5}{9}, \frac{5}{12} \ldots\right\}$

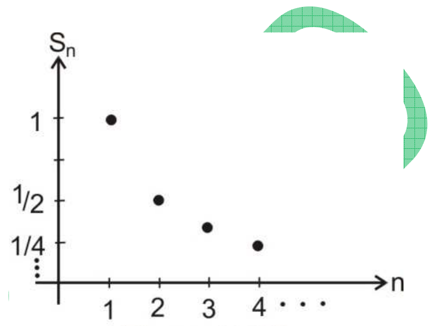

Figure 3.38

LIMIT OF A SEQUENCE

If the nth term S_n of a sequence approaches to "l" as n approaches infinity then "l" is called limit of the sequence.

$$\lim_{n\to\infty} s_n = l$$

Example 3.13:

The nth term and the limit of some sequences are given below:

$(i)\ \ S_n = \frac{3n}{n+1}$; $\lim_{n\to\infty} s_n = 3$

$(ii)\ \ S_n = \frac{n^2}{n^2+9}$; $\lim_{n\to\infty} s_n = 1$

$(iii)\ S_n = \frac{5n+3}{3n+2}$; $\lim_{n\to\infty} s_n = \frac{5}{3}$

CONVERGENT SEQUENCE:

A sequence $\{S_n\}$ converges to l if for every $\varepsilon > 0$, there exists a positive integer $\mathbb{N}$ depends on ε such that

$$|S_n - l| < \varepsilon \quad , \forall \; n \geq \mathbb{N}$$

or $$\lim_{n\to\infty} S_n = l$$

To understand it consider the **figure 3.32(a).**

The graph of S_n shows that the values of S_n lies between $l - \varepsilon$ and $l + \varepsilon$ for all $n \geq \mathbb{N}$, or the distance between S_n and l less than ε for all $n \geq N$.

$$|s_n - l| < \varepsilon \quad \text{for all} \quad n \geq \mathbb{N}.$$

N depends on ε:

If we suppose $\varepsilon_1 > 0$ greater than ε then the value of N_1 will be less than or equal to N as shown in the **figure 3.32(b).**

$$|s_n - l| < \varepsilon_1 \quad \text{for all} \quad n \geq N_1 \text{ such that } N_1 < N$$

Hence S_n converges to l.

If we suppose $\varepsilon_2 > 0$ less than ε then the value of N_2 will be greater than or equal to N as shown in the **figure 3.32(c).**

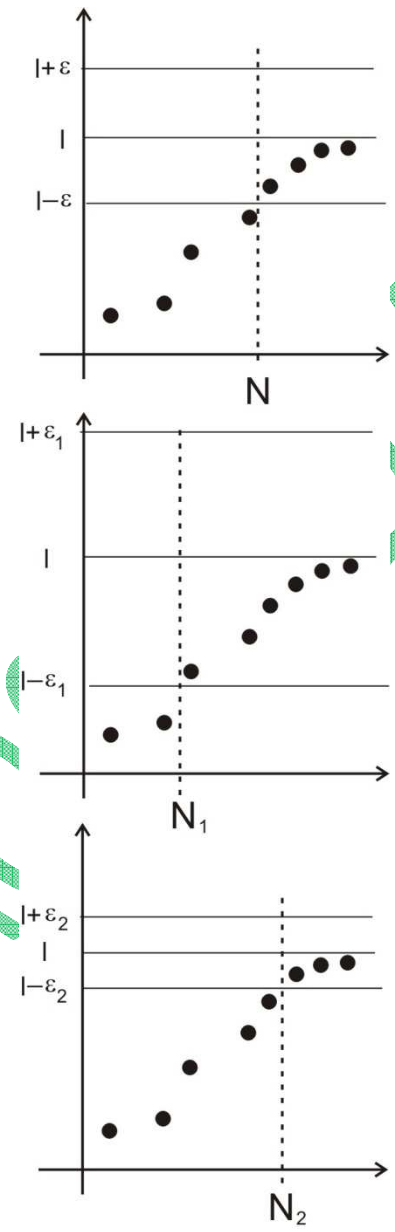

Figure 3.39 *a, b,c*

Examples 3.14:

Consider the nth term of a sequence

$$S_n = \frac{3n}{n+10}$$

$$\lim_{n\to\infty} S_n = 3$$

S_n is a convergent sequence.

NULL SEQUENCE:

A sequence is called null sequence if it converges to zero.

$$\lim_{n\to\infty} S_n = 0$$

Theorem A-1:

The limit of a convergent sequence is unique.

Proof:

Let $\{S_n\}$ be a convergent sequence which converges to two different limits l and l' then

$$|\, S_n - l \,| < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_1$$

and $$|\, S_n - l' \,| < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_2$$

On adding

$$|\, S_n - l \,| + |\, S_n - l' \,| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N$$

$| S_n - l | + | l' - S_n | < \varepsilon$ for all $n \geq N$
where $N = \max(N_1, N_2)$
$| (S_n - l) + (l' - S_n) | < | Sn - l | + | l' - S_n | < \varepsilon$
$| Sn - l - Sn + l' | < \varepsilon$ for all $n \geq N$
$|l' - l| < \varepsilon$ for all $n \geq N$
which is a contradiction.
Hence the limit of a convergent sequence is unique.

Theorem A-2:
A continuous function f in a closed interval $[a, b]$ has greater and least values.

Proof:
We have to prove that the set of the values of $f(x)$ for all $x \in [a, b]$ has greatest lower bound m and least upper bound M.
Since f is continuous on $[a, b]$ hence f is bounded.
We consider the case of least upper bound M.
Suppose that $M \notin \{f(x)\}$ for all $x \in [a, b]$
$\Rightarrow \quad M > f(x)$ for all $x \in [a, b]$
$\Rightarrow \quad M - f(x) \neq 0$ for all $x \in [a, b]$
$\Rightarrow \quad \dfrac{1}{M - f(x)}$ is continuous on $[a, b]$

$\Rightarrow \quad \dfrac{1}{M - f(x)}$ is bounded.
Since M is supremum, hence for a given real number c there exists at least one $x_r \in [a, b]$ such that
$f(x_r) > M - \frac{1}{c}$ by the definition of supremum.
$\Rightarrow \quad \dfrac{1}{M - f(x_r)} > c$
$\frac{1}{M - f(x_r)}$ is not bounded because c is a given number.
It is a contradiction.
Hence f attains its supremum or $\in \{f(x)\}$.
Similarly we can prove for infimum.

EXERCISE 3

Show that following sequences converge to indicated number.

(1) $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$; to zero

(2) $a_n = \frac{1}{n^2}(1 + 2 + 3 + \cdots + n)$; to $\frac{1}{2}$

(3) $a_n = \frac{1}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3)$; to $\frac{1}{4}$

(4) Find the upper bounds, lower bounds, least upper bound and greatest lower bound of the following sets:

(i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right\}$; $A \subset R$

(ii) $B = \{\frac{1}{n^2}(1 + 2 + 3 + \cdots + n) \mid n \in N\}$; $B \subset R$

(iii) $S = \{\frac{1}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3) \mid n \in N\}$; $S \subset R$

(5) If $\{a_n\}$ and $\{b_n\}$ are null sequences then show that the following sequences are also null sequences.

(i) $\{Ka_n\}$; K is a real constant.

(ii) $\{a_n + b_n\}$

(iii) $\{f_n a_n\}$; $\{f_n\}$ is bounded.

(6) If $\{a_n\}$ converges to "a" and $\{b_n\}$ converges to "b" then show that

(i) $\{a_n - b_n\}$ converges to $a + b$

(ii) $\{a_n b_n\}$ converges to ab

(iii) $\left\{\frac{1}{a_n}\right\}$ converges to $\frac{1}{a}$

(iv) $\left\{\frac{a_n}{b_n}\right\}$ converges to $\frac{a}{b}$, $b \neq 0$