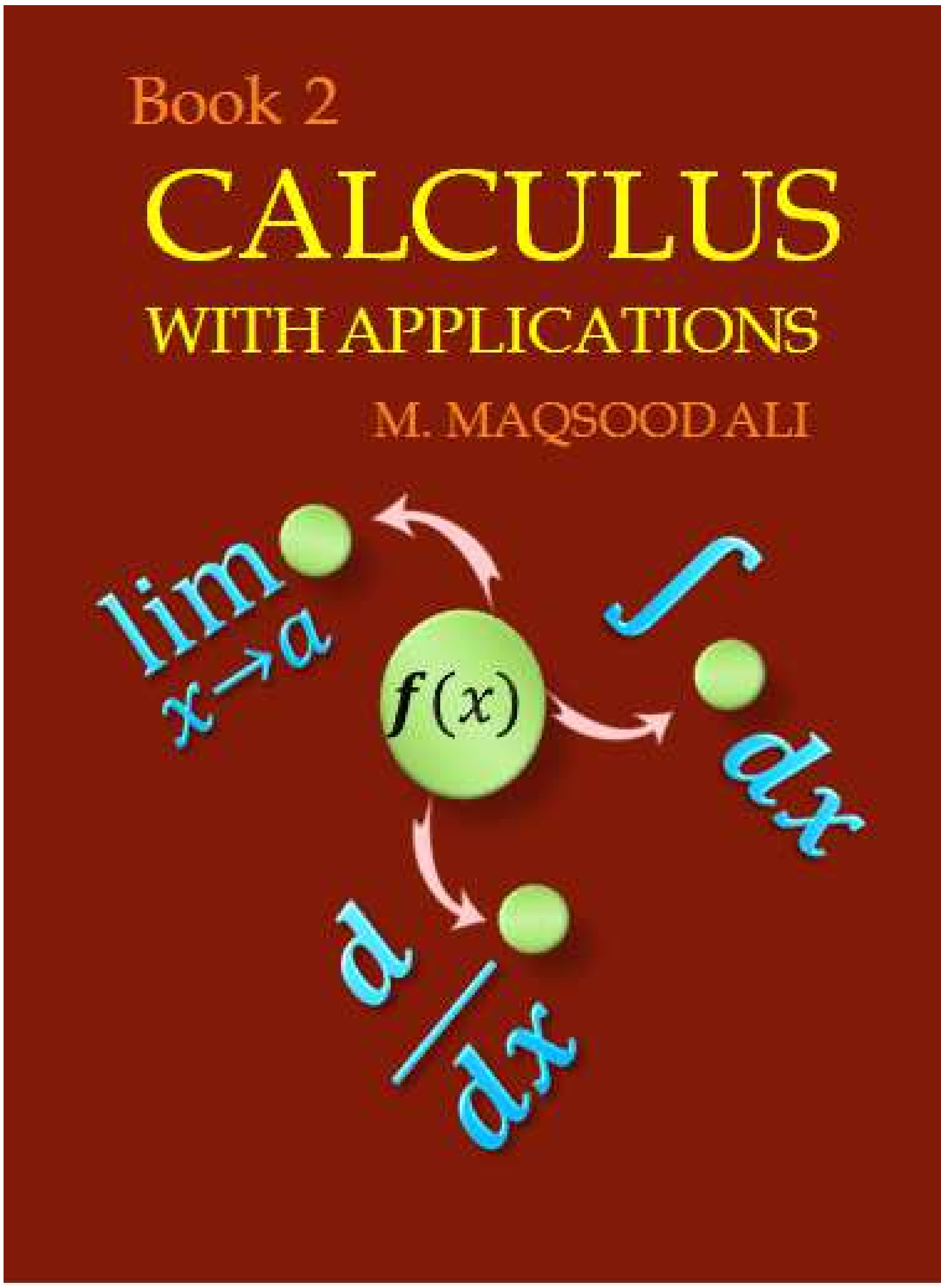

Book 2
CALCULUS
WITH APPLICATIONS
M. MAQSOOD ALI
$\lim_{x \to a}$
$f(x)$
$\int dx$
$\frac{d}{dx}$

FUNCTIONS FORMING USING THE GRAPHS OF DIFFERENT FUNCTIONS:

The graphs of many different functions are given. To form a design using some graphs a function can be formed combining these functions of the graphs. When the function is formed the values of arbitrary constants in the functions are written according to the design or map.

Example 2.22:

The map of front side of a house is designed using the curves as shown in the **figures 2.38, 2.39** and straight lines.

Find the following function $f(x)$ which represents the map, as shown in **figure 2.40**.

$$f(x)=\begin{cases} a+\cos x \\ b+\sin(x-c) \\ d \end{cases}$$

(**Hint:** Find the values of a, b, c and d.

Solution:

The graph $g(x)$ is formed shifting $\cos x$ vertically 12 units, so that

$$g(x)=12+\cos x$$

The graph $h(x)$ is formed shifting the curve $\sin x$, horizontally and vertically $\frac{\pi}{2}$ units and 7 units respectively, so that

$$h(x)=7+\sin(x-\frac{\pi}{2})$$

The graph $k(x)$ is a straight line parallel to $x-axis$ and $y-$ intercept zero, so that

$$k(0)=0$$

The function $f(x)$ is

$$f(x)=\begin{cases} 12+\cos x & \text{for} \quad 0\le x\le 2\pi \\ 7+\sin(x-\pi/2) & \text{for} \quad \pi/2\le x\le 3\pi/2 \\ 0 & \text{for} \quad 0\le x\le 2\pi \end{cases}$$

Physical Science:

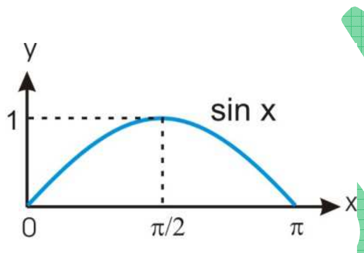


Figure 2.38

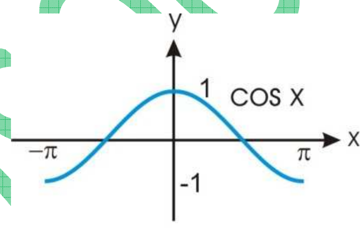


Figure 2.39

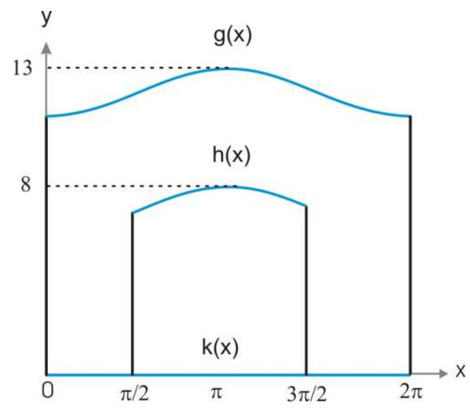


Figure 2.40

Example 2.23:

Physical Science:

A solid of revolution, **figure 2.45**, is created by revolving the region bounded by the graph $f(x)$ and $x-$ axis, **figures 2.44** and 2.45. The graph $f(x)$ is form by the curves and straight lines as shown in **figures 2.41, 2.42** and **2.43**.
The graph of the function $f(x)$ is shown in figure 2.44. Find the function $f(x)$.
Solution:
$g(x)$ is a straight line parallel to $x-axis$ and $y-$ intercept 2, so that

$$g(x)=2$$

$h(x)$ is the curve formed by shifting the curve $\frac{sinx}{x}$ vertically and horizontally 1 unit and 3 units respectively, so that

$$h(x)=1+\frac{\sin(x-3)}{(x-3)}$$

$k(x)$ is a straight line joining the points $(9.3, h\,(9.3))$ and $(13,0)$

$$h\,(9.3)=1+\frac{\sin 6.3}{6.3}=1+0=1$$

the slope of the line is

$$m=\frac{0-1}{13-9.3}=-\frac{1}{3.7}=-\frac{10}{37}$$

The equation of the line is

$$y=-\frac{10}{37}x+\frac{130}{37}$$

or

$$k(x)=-\frac{10}{37}x+\frac{130}{37}$$

The function $f(x)$ is

$$f(x)=\begin{cases} 2 & for \quad 0\le x\le 3 \\ 1+\dfrac{\sin(x-3)}{(x-3)} & for \quad 3<x\le 9.3 \\ -\dfrac{10}{37}x+\dfrac{130}{37} & for \quad 9.3<x\le 13 \end{cases}$$

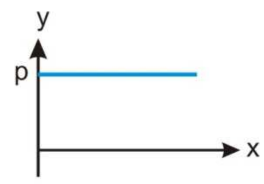


Figure 2.41

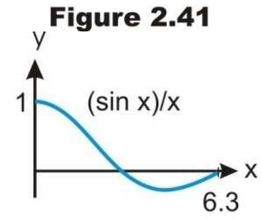


Figure 2.42

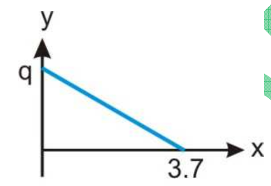


Figure 2.43

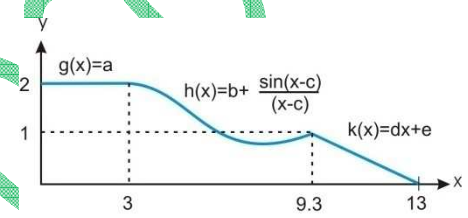


Figure 2.44

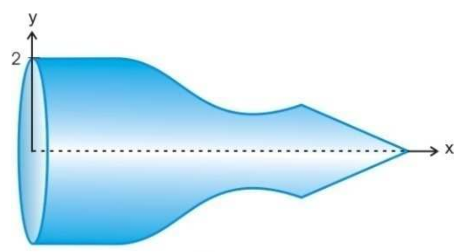
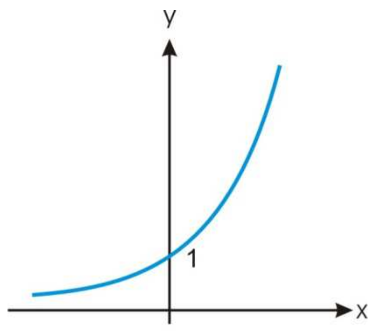
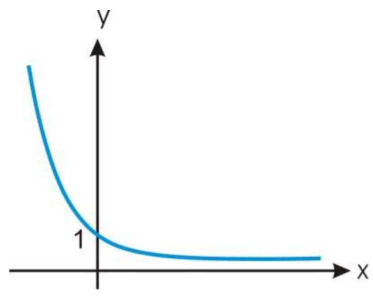


Figure 2.45

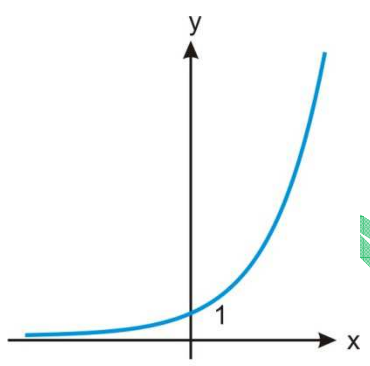
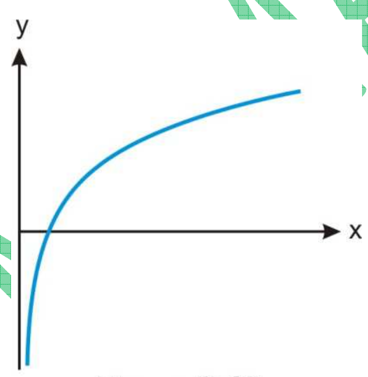
EXPONENTIAL FUNCTIONS:
A function of the form

$$f(x) = b^x$$

is an exponential function, where b is a positive real number is the base of the function and independent variable x is in the exponent.

$b > 1$	$0 < b < 1$
 **Figure 2.46** The function $f(x) = b^x$ is increasing function.	 **Figure 2.47** The function $f(x) = b^x$ is decreasing function.

Natural Exponential and Logarithmic Functions:

Natural Exponential Function	Natural Logarithmic Function
$f(x) = e^x$  **Figure 2.48**	$f(x) = lnx$ or $f(x) = log_e x$  **Figure 2.49**

The value of e:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

HOW AND WHEN TO FORM AN EXPONENTIAL FUNCTION:

Suppose that f is a function whose rate of change is directly proportional to its present value.

$$\frac{d}{dt}f(t) \propto f(t)$$

$$\frac{d}{dt}f(t) = Kf(t)$$

It is a differential equation of order one, where K is a constant of proportionality.

$$\int \frac{1}{f(t)}df(t) = \int K\,dt$$

$$lnf(t) = Kt + C \qquad \rightarrow (1)$$

At $t = 0$, $f(t) = f(t_0)$

$$\ln f(t_0) = 0 + C$$

$$C = \ln f(t_0)$$

Putting in (1)

$$\ln f(t) = Kt + \ln f(t_0)$$

$$\ln f(t) - \ln f(t_0) = Kt$$

$$\ln \frac{f(t)}{f(t_0)} = Kt$$

$$\frac{f(t)}{f(t_0)} = e^{Kt}$$

$$f(t) = f(t_0)\,e^{Kt}$$

Note:

In this case , when

$$\frac{d}{dt}f(t) = Kf(t)$$

The function $f(t)$ is

$$f(t) = f(t_0)\ e^{Kt}$$

Example 2.24:

Physical Science:

A man wants to cut a sheet of paper into equal small pieces. He cuts the sheet into two equal pieces, and then two pieces into four, four pieces into eight and so on.

(a) Find the number of pieces as a function of number of attempt.

(b) Find number of attempts as a function of number of pieces of the sheet.
(c) How many pieces of paper will he have after 9th attempt?
(d) How many attempts are required for 1024 pieces of papers?

Solution:

(a) Number of attempt$= x$
Number of pieces$= A(x)$

x	0	1	2	3	...
$f(x)$	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	...

The function A is

$$A(x) = 2^x$$

Figure 2.50

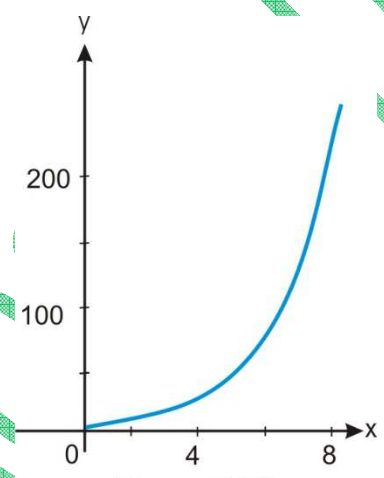


Figure 2.50

(b) Number of attempt$= x_{(A)}$
Number of pieces$= A$

$$A = 2^x$$

$$x_{(A)} = \log_2 A$$

(c) $x = 9$

$$A(9) = 2^9 = 512$$

He has 512 pieces of papers after 9th attempt.

(d)

$$x(1024) = \log_2 1024 = \frac{\log_{10} 1024}{\log_{10} 2} = 10$$

10 attempts are required.

COMPOUND INTEREST (FORMULA):

The interest is called compounded if the interest is added to the principal at the end of each interest period. So that

current principal = last principal + interest

Formula:

(i) $A = P(1 + r)^n$
(ii) Compound interest = Total amount − Principal

$$= A - P$$

where

P : Principal

r : interest rate per time period

n : Number of time periods

MULTIPLE COMPOUNDING (FORMULA):

The interest may be compounded annually , semi annually, quarterly, monthly, daily and so on. Usually the rate of compound specified as an annual rate called nomial rate. Let

i : Rate of interest per year

m : Number of time periods in a year, the interest is compounded

t : Number of years

$$A = P(1 + r)^n$$

Putting $r = \frac{i}{m}$ and $n = mt$

$$A = P\left(1 + \frac{i}{m}\right)^{mt}$$

Total amount as a function of t is

$$A(t) = P\left(1 + \frac{i}{m}\right)^{mt}$$

CONTINUOUSLY COMPOUNDED INTEREST:

The compounded interest

$$A(t) = P\left(1 + \frac{i}{m}\right)^{mt}$$

$$A(t) = P\left\{\left(1 + \frac{i}{m}\right)^{m}\right\}^{t}$$

The compounding is occuring all the time because the compounding is continuous, so that

Number of time periods in a year are infinite,

so $m \to \infty$

$$\lim_{m\to\infty}\left(1 + \frac{i}{m}\right)^{m} = e^{i}$$

So that

$$A(t) = Pe^{it}$$

Example 2.25: **Business and Economics**

5000 dollars are invested at an annual interest rate 8 percent. Compute the balance at the end of 5 years, if the interest is compounded continuously.

Solution:

$P = \$\,5000$ and $r = 8\% = 0.08$

$$A(t) = Pe^{it}$$
$$A(t) = 5000e^{0.08t}$$

Figure 2.51.

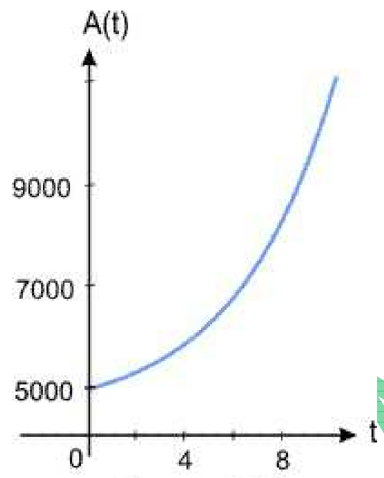


Figure 2.51

Putting $t = 5$

$$A(5) = 5000\, e^{0.08(5)}$$
$$= 74591.23$$

The balance at the end of 5 years is 74591.23 dollars.

Example 2.26:

A laboratory starts with an initial bacterial population of 6 grams. The population grows at a rate of propositional to its size, doubling every 48 hours. How much time is required for 80 grams of bacteria?

Solution:

Life Science

Population of bacteria at time $t = P(t) = ?$

Starting population of bacteria $= P(0) = P_0 = 6$ grams

The rate of growth is proportional to the size of population, so that the differential equation is

$$\frac{dp}{dt} = KP$$

Where K is a constant of proportionality. The solution of differential equation is

$$P(t) = P_0\, e^{Kt}$$
$$P(t) = 6\, e^{Kt} \quad \rightarrow \quad (1)$$

The population doubles every 48 hours, so that

$$P(48) = 12 \text{ grams}$$

Putting in (1)

$$P(48) = 6\, e^{48K}$$
$$12 = 6\, e^{48K}$$
$$e^{48K} = 12$$
$$48K = \ln 2$$
$$K = 0.0144$$

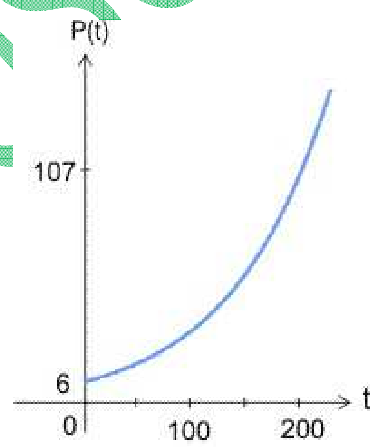


Figure 2.52

Putting in (1) , we have

$$P(t) = 6\, e^{0.0144t}$$

Is the population of bacteria at time t, **figure 2.52.**

$P(t) = 80$ grams

$$80 = 6\, e^{0.0144t}$$
$$e^{0.0144t} = 13.33$$
$$t = \frac{\ln 13.33}{0.0144} = 179.86 \text{ hours}$$

Example 2.27:

A half life of radioactive carbon-14 is 5730 years. How much of the 500 grams carbon-14 would be left after 2000 years?

Lile Science

Solution:

The decay of radioactive carbon-14 is an exponential process. The amount (t) of carbon -14 after t years is

$$A(t) = A_0\, e^{kt} \quad \rightarrow \quad (1)$$

Firstly calculate the value of k using half life of carbon-14.

$$A(5730) = \frac{1}{2}\, A_0$$

Using (1)

$$A(5730) = A_0\, e^{5730k}$$

$$\frac{1}{2} A_0 = A_0\, e^{5730k}$$

$$0.5 = e^{5730K}$$

$$5730k = \ln 0.5$$
$$k = -0.000121$$

So that

$$A(t) = A_0\, e^{-0.000121t}$$

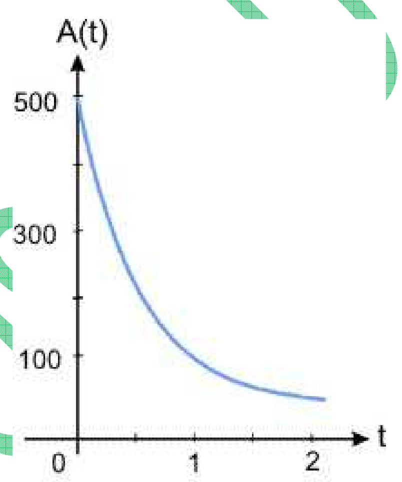


Figure 2.53.

Putting $A_0 = A(0) = 500$ grams

$$A(t) = 500\, e^{-0.000121t}$$

Putting $t = 2000$ years

$$A(2000) = 500\, e^{-0.000121(2000)}$$
$$= 392.53 \text{ grams.}$$

AGE OF FOSSILS OR DATE OF DEATH:

Carbon-14 is a radioactive isotope of carbon. It is used to determine the age of fossils or date of death of an animal or a plant from a few years old to 50,000 years old. This method to estimate the age of organic material based on the decay rate of carboun-14. The half life of radioactive carbon-14 is 5730 years. Carbon-14 is produced by the interaction of the cosmic rays on nitrogen in the atmosphere.

$${}^{14}_{7}N + {}^{1}_{0}n \rightarrow {}^{14}_{6}C + {}^{1}_{1}H$$

The process has been going on for thousand of years, so that the ratio of carbou-14 to carbon-12 in the atmosphere is relatively constant. Living plants and animals absorb the carbon from the air and the food. The ratio of carbon-14 and carbon-12 in the living plants and animals is same as in the atmosphere. When that plant or animal dies, it stops intaking carbon as food or air, the carbon-14 begin to decay but the carbon-12 does not decay.

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + {}^{0}_{-1}n$$

By measuring how much the ratio is lowered, the scientists can determine the date of death.

Example 2.28:

An animal's bone is discovered. The bone contains one-seventh the expected amount of carbon 14. How old is the discovered bone? The half-life of carbon 14 is 5730 years. **Life Science**

Solution:

The rate of decay is proportional to the amount A of the carbon 14, so that the differential equation is

$$\frac{dA}{dt} = kA \qquad \rightarrow (1)$$

Where k is a constant of proportionality. The solution of differential equation is

$$A(t) = A_0\, e^{kt} \qquad \rightarrow (2)$$

Firstly calculate the value of k using half life of carbon-14.

$$A(5730) = \frac{1}{2} A_0 \qquad \rightarrow (3)$$

PARTICULAR FUNCTIONS

MODULUS FUNCTION:

The modulus function is written as

$$f(x) = \begin{cases} -x & \text{for} \quad x < 0 \\ x & \text{for} \quad x \geq 0 \end{cases}$$

Figure 2.55.

For example

$|-5| = -(-5) = 5$ and $|6| = 6$

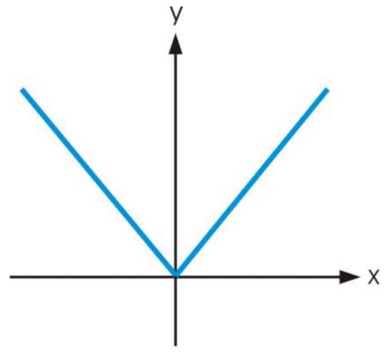


Figure 2.55

Business and Economices

Example 2.29:

A business man gets 12% profit on the amount which he invests. He invests amount which he has and sometimes by taking loan from the bank. Find the total amount after getting profit as a function of amount he invests.

(a) Calculate the total amount after getting profit on \$ 5000.

(b) Calculate the total amount after getting profit, if the invested amount is taken loan \$ 4000 from the bank.

Solution:

Let x be the invested amount and $f(x)$ total amount after adding profit 12%

$$f(x) = x + \frac{12}{100}|x|$$
$$f(x) = x + 0.12\,|x|$$

(a) $x = \$\,5000$

$$f(5000) = 5000 + 0.12\,|5000|$$
$$= 5000 + 600$$
$$= 5600$$

Total amount is 5600 dollars.

(b) $x = -4000$

Where negative sign shows that the invested amount is a loan.

$$f(-4000) = -4000 + 0.12\,|-4000|$$
$$= -4000 + 0.12(4000)$$
$$= -3520.$$

The reaming loan is \$ 3520.

GREATEST INTEGER FUNCTION: OR BRACKET FUNCTION [x]:

The greatest integer function is

$$f(x) = \lfloor x \rfloor$$

where, $\lfloor x \rfloor =$ the largest integer $\le x$.

It can be defined as

$$\lfloor x \rfloor = a \quad \text{for} \quad a \le x < a + 1$$

where a is an integer. So that

$\lfloor x \rfloor = -2$	for	$-2 \le x < -1$
$\lfloor x \rfloor = -1$	for	$-1 \le x < 0$
$\lfloor x \rfloor = 0$	for	$0 \le x < 1$
$\lfloor x \rfloor = 1$	for	$1 \le x < 2$
$\lfloor x \rfloor = 2$	for	$2 \le x < 3$

Figure 2.57.

For example
$\lfloor 2.96 \rfloor = 2, \quad \lfloor 2.02 \rfloor = 2, \quad \lfloor -8.67 \rfloor = -9$
Graph of the greatest inter function $f(x) = \lfloor x \rfloor$ is shown in **figure 2.57**.

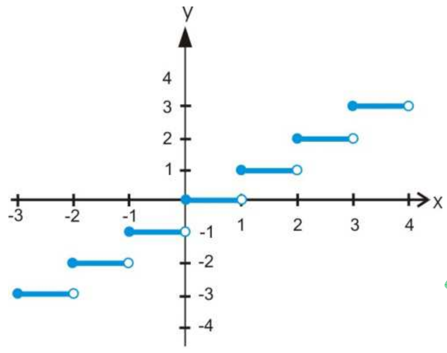


Figure 2.57

Example 2.30:

Business and Economics

A man deposits \$ 8000 in a bank. The interest \$ 70 is added at the beginning of each new month. Find the Amount as a function of number of months.

(a) Draw the graph.
(b) Find the amount at the end of 2.5 months.
(c) Find the total amount at the end of 2.25 months.

Solution:

Let x be the number of months and f(x) total amount. The amount at any day can be calculated by the function.

$$f(x) = 8000 + 70 \lfloor x \rfloor$$

(a) The graph of $f(x)$ against x is shown in **figure 2.58**.

(b) $x = 2.5$

$$f(2.5) = 8000 + 70 \lfloor 2.5 \rfloor$$
$$= 8000 + 70(2)$$
$$= 8140$$

Total amount at the end of 2.5 months is 8140 dollars.

(c) $x = 2.25$

$$f(2.25) = 8000 + 70 \lfloor 2.25 \rfloor$$
$$= 8000 + 70(2)$$
$$= 8140$$

Total amount at the end of 2.25 months is 8140 dollars.

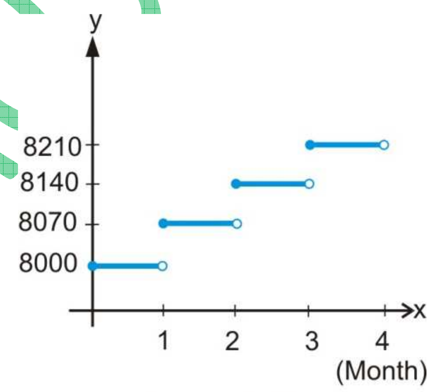


Figure 2.58

LEAST INTEGER FUNCTION:

The least integer function is written as

$$f(x) = \lceil x \rceil$$

where

$\lceil x \rceil$ = the least integer $\geq x$

It can be defined as

$\lceil x \rceil = a + 1 \quad$ for $\quad a \leq x < a + 1$

where a is an integer. So that

$\lceil x \rceil = -2 \quad$ for $\quad -3 < x \leq -2$
$\lceil x \rceil = -1 \quad$ for $\quad -2 < x \leq -1$
$\lceil x \rceil = 0 \quad$ for $\quad -1 < x \leq 0$
$\lceil x \rceil = 1 \quad$ for $\quad 0 < x \leq 1$
$\lceil x \rceil = 2 \quad$ for $\quad 1 < x \leq 2$

Figure 2.59.

For example:

$\lceil 1.35 \rceil = 2$, $\lceil 1.02 \rceil = 2$, $\lceil -3.21 \rceil = -2$

Graph of the least integer function $f(x) = \lceil x \rceil$ is shown in **figure 2.59**.

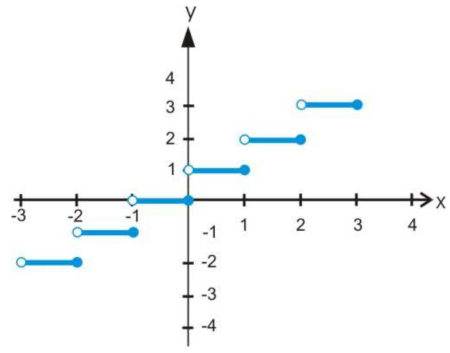


Figure 2.59

Example 2.31:

The bus fare in a city is ¢ 10 plus ¢ 2 per kilometer. Find the fare as a function of distance in kilometers.

(a) Draw the graph of the function.

(b) Calculate the fare to travel 2km,2.2km, 2.9km.

Solution:

Let x be the distance in kilometers and $f(x)$ fare of the bus in cents. The function $f(x)$ is

$$f(x) = 10 + 2\lceil x \rceil$$

(a) The graph of the function is shown in **figure 2.60**.

(b) $x = 2, 2.2$ and 2.9

$f(2) = 10 + 2\lceil 2 \rceil = 10 + 2(2) = 14$ cents

$f(2.2) = 10 + 2\lceil 2.2 \rceil = 10 + 2(3) = 16$ cents

$f(2.9) = 10 + 2\lceil 2.9 \rceil = 10 + 2(3) = 16$ cents

The fare to travel $2km$, .$2.2km$ and 2.9 km are ¢ 14 ,16 and 16 cents respectively.

Business and Economices

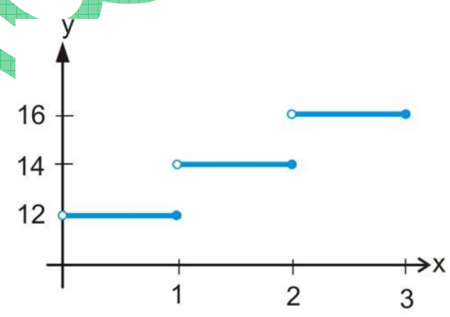


Figure 2.60