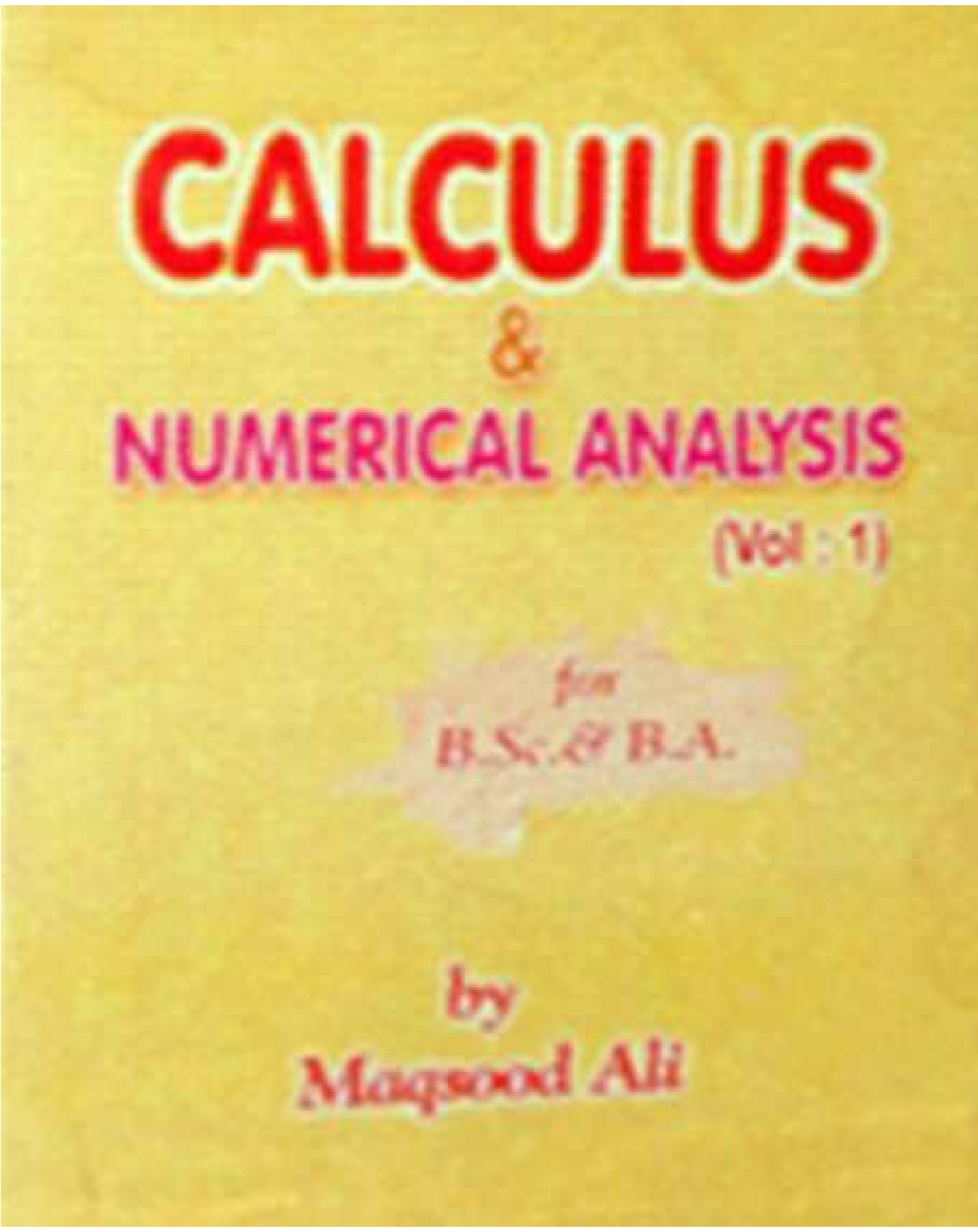
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
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SYSTEM OF LINEAR EQUATIONS

System of linear equations can be solved by the following methods.
(i) Jacobi iteration method
(ii) Gauss seidel iterative method
(iii) Relaxation method

(i) Jacobi Iteration Method:

Consider the system of m linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2m}x_m &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mm}x_m &= b_m \end{aligned}$$

or
$$\sum_{j=1}^{m} a_{ij}x_j = b_i \qquad , \qquad i = 1,2,3 \cdots , m$$

To find the values of m variables by Jacobi iterative method rearrange the rows so that the diagonal elements have magnitude as large as possible relative to the magnitudes of other coefficients in the same row.

If
$$|a_{ii}| > \sum_{\substack{i=1 \\ j\neq i}}^{m} |a_{ij}| \quad , \qquad i = 1, 2, 3, \cdots , m$$

Then $x^{(n)}$ will converge to the solution whether any initial values are used.

Write the equations as given below

$$x_i^{(n+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{a_{ij}}{a_{ii}} x_j^{(n)}$$

To solve the following system of equations

$$\begin{aligned} x_1 + 5x_2 - x_3 &= 10 \\ x_1 + x_2 + 8x_3 &= 20 \\ 4x_1 + 2x_2 + x_3 &= 14 \end{aligned}$$

Rearrange the equations as

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 14 \\ x_1 + 5x_2 - x_3 &= 10 \\ x_1 + x_2 + 8x_3 &= 20 \end{aligned}$$

Divide $1st$,$2nd$ and $3rd$ equations by the coefficients of x_1, x_2, x_3 respectively, and arrange as given below.

$$\begin{aligned} x_1 &= 3.5 - 0.5x_2 - 0.25x_3 \\ x_2 &= 2 - 0.2x_1 + 0.2x_3 \\ x_3 &= 2.5 - 0.125x_1 - 0.125x_2 \end{aligned}$$

$1st$,$2nd$ and $3rd$ equations are used for the values of x_1, x_2,and x_3.

$x_1 = 0, x_2 = 0, x_3 = 0$ are the starting values.

To find the values of $x_1, x_2, \cdots, x_m$. We begin with initial approximate values of x_1, x_2,and x_3. If there are not initial approximate values then we can begin with $x_1 = 0, x_2 = 0, x_3 = 0$.

By putting these values in right hand sides of the set of equations we get new approximate values $3.5, 2.0, 2.5$ closer to the true values. Now we put these new approximate values into the right hand sides of the set of equations to get new approximate values $1.87, 1.8, 1.81$ which are closer to the true values. The process is repeated until the values of each of the variables are sufficiently alike as shown below.

Variables	I	II	III	IV	V	VI	VII	VII	IX
x_1	0	3.5	1.85	2.14	1.99	2.01	2.00	2.00	2.00
x_2	0	2.0	1.8	1.98	1.97	1.99	1.99	1.99	2.00
x_3	0	2.5	1.81	2.04	1.98	2.00	1.99	1.99	2.00

Hence $x_1 = 2, x_2 = 2,\ x_3 = 2.$

(ii) GAUSS-SEIDEL ITERATIVE METHOD:

The only difference of Jacobi method and Gauss-Seidel method is that in Gauss-Seidel method we use new value of $x_1(i.e. x_1^{(n+1)}$ and x_2 and we used new values of $x_1(i.e. x_1^{(n+1)})$ and $x_2\ (i.e. x_2^{(n+1)})$ to find x_3, as shown in the following equations.

The above equations can be written as

$$x_1^{(n+1)} = 3.5 - 0.5\, x_2^{(n)} - 0.25x_3^{(n)}$$
$$x_2^{(n+1)} = 2 - 0.2\, x_1^{(n+1)} + 0.2x_3^{(n)}$$
$$x_3^{(n+1)} = 2.5 - 0.125x_1^{(n+1)} - 0.125x_2^{(n+1)}$$

Variables	I	II	III	IV	V	VI	VII
x_1	0	3.5	2.375	2.056	2.010	2.001	2.000
x_2	0	1.3	1.905	1.982	1.997	1.999	2.000
x_3	0	1.9	1.965	1.995	1.999	1.999	2.000

Hence $x_1 = 2, x_2 = 2,\ x_3 = 2.$

EXERCISE D-9

(1) $6x + 2y - 3z = 11$
$3x - 2y + 6z = 16$
$x + 9y - 7z = 17$

(2) $3x - y + 5z = 62$
$2x + 5y + z = 36$
$7x - 3y + z = 14$

(3) $9x + 17y + 30z = 584$
$7x + 15y + z = 20$
$5x + 2y + 2z = -21$

(4) $20x + 5y + z = -9$
$4x + 8y + 20z = 32$
$x + 4y + 2z = 9$

(5) Solve $Ex.3$ with starting values $5, 10, 15$.

RELAXATION METHOD

How to solve system of linear equations by "relaxation method" consider the following system of equations.

$$x_1 + 5x_2 - x_3 = 10$$
$$x_1 + x_2 + 8x_3 = 20$$
$$4x_1 + 2x_2 + x_3 = 14$$

I- Transpose all the terms to one side.

$$x_1 + 5x_2 - x_3 - 10 = 0$$
$$x_1 + x_2 + 8x_3 - 20 = 0$$
$$4x_1 + 2x_2 + x_3 - 14 = 0$$

II- Rearrange these equations by the largest coefficient of x_1, x_2 and x_3 , this arrangement is different from Jacobi method.

$$4x_1 + 2x_2 + x_3 - 14 = 0$$
$$x_1 + 5x_2 - x_3 - 10 = 0$$
$$x_1 + x_2 + 8x_3 - 20 = 0$$

$$R_1 = 0,\ R_2 = 1300,\ R_3 = 2062$$

The largest residual is $R_3 = 2062$.

IX- To make the values of $R_3 = 2062$ zero multiply 2062 by the coefficient of x_3 and add in R_1, R_2 and R_3.

X- For $x_2 = 2062, x_1 = 0, x_3 = 0$ we get

$$R_1 = -516,\ R_2 = 1712,\ R_3 = 0$$

XI- We continue this process select new residual of greatest magnitude relax it to zero, until all residual are zero. We add all the values in columns of x_1, x_2,and x_3 and dividing by 1000 we get the solution.

x_1	x_2	x_3
-1	-0.5	-0.25
-0.2	-1	$+0.2$
-0.125	-0.125	-1

x_1	R_1	x_2	R_2	x_3	R_3
0	3500	0	2000	0	2500
	-3500		-700		-438
3500	0		1300		2062
	-516		412		-2062
	-516		1712	2062	0
	-856		-1712		-214
	-1372	1712	0		-214
	1372		274		172
-1372	0		274		-42
	-137		-274		-34
	-137	274	0		-76
	137		27		17
-137	0		27		-59
	15		-12		59
	15		15	-59	0
	-15		-3		-2
	0		12		-2
	-6		-12		-2
	-6	12	0		-4
	6		1		1
	0		1		-3
	0.1		-1		3
	1		0		0
	-1		0	-3	0
	0		0		0
2001		1998		2000	

$x_1 = 2$, $x_2 = 2$, $x_3 = 2$

EXERCISE D-10

Solve the following by Relaxation method .

(1) $5x - y - 4z + 8 = 0$
$x + 3y + z + 5 = 0$
$2x + y - 3z + 3 = 0$

(2) $2x + 3y + z = 5$
$5x + y + 2z = 8$
$x + y + 3z = 6$

(3) $x + 5y - z = 10$
$x + y + 8z = 20$
$4x + 2y + z = 14$

(1) $10x + 5y - 8z + 15 = 0$
$x - 4y + 5z + 10 = 0$
$-2x + y - 5z + 4 = 0$

INTERPOLATION

In this section we will study that how can construct polynomial function by the given values of $f(x)$ at some point $x \in D_f$.

The given values of $f(x)$ shown in the following table.

x	x_1	x_2	x_3	x_4	$\cdots$	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$\cdots$	$f(x_n)$

LAGRANGE INTERPOLATION:

The lagrange polynomial may be generalized to the nth order as given below.

$$P(x) = \sum_{i=1}^{n+1} f(x_i) \prod_{\substack{j\neq i \\ j=1}}^{n+1} \frac{(x - x_i)}{(x_i - x_j)}$$

Where n is the order of the polynomial.

It is simple to program on a computer but extremely cumbersome to use for hand calculation.

Example 14:

Find a polynomial by lagrange interpolation,for the following data.

(i) $(x_1 f(x_1))\,, (x_2 f(x_2))\,, (x_3 f(x_3))\,,$

(ii) $(1,2), (3,5), (6,8)$

Also find the values of $f(3)$ and $f(4)$.

Solution:

(i) The given values are

x	x_1	x_2	x_3
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

$$f(x) = \sum_{i=1}^{2+1} f(x_i) \prod_{\substack{j\neq i \\ j=1}}^{2+1} \frac{(x - x_i)}{x_i - x_j}$$

$$= \sum_{i=1}^{3} f(x_i) \prod_{\substack{j\neq i \\ j=1}}^{3} \frac{(x - x_i)}{x_i - x_j}$$

$$= f(x_1)\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2)\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + f(x_3)\frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

(ii) The given values are

$\big(x_1, f(x_1)\big) = (1,2)\ ,\ \big(x_2\, , f(x_2)\big) = (3,5)\ ,\ \big(x_3\, , f(x_3)\big) = (6,8)$

$$f(x) = \sum_{i=1}^{2+1} f(x_i) \prod_{\substack{j \neq i \\ j=1}}^{2+1} \frac{(x - x_i)}{x_i - x_j}$$

$$= \sum_{i=1}^{3} f(x_i) \prod_{\substack{j \neq i \\ j=1}}^{3} \frac{(x - x_i)}{x_i - x_j}$$

$$f(x) = f(x_1)\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2)\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + f(x_3)\frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

$$f(x) = \frac{2(x - 3)(x - 6)}{(1 - 3)(1 - 6)} + \frac{5(x - 1)(x - 6)}{(3 - 1)(3 - 6)} + \frac{8(x - 1)(x - 3)}{(6 - 1)(6 - 3)}$$

$$f(x) = \frac{1}{30}(-3x^2 + 57x + 6)$$

Now we find the values of $f(3)$ and $f(4)$

$$f(3) = \frac{1}{30}\{-3(3)^2 + 57(3) + 6\} = 5$$

$$f(4) = \frac{1}{30}\{-3(4)^2 + 57(4) + 6\} = 6.2$$

DIVIDED DIFFERENCE TABLES:

The given values of $f(x)$ shown in the following table.

x	x_1	x_2	x_3	x_4	$\cdots$	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$\cdots$	$f(x_n)$

The polynomial function for the above data is

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_n)$$

The values of $a_0, a_1, \cdots, a_n$ can be found by the divided difference table.

DIVIDED DIFFERENCE TABLES (Unequal Intervals):

x_i	$a_0 = f$	$a_1 = \Delta f$	$a_2 = \Delta^2 f$	$a_3 = \Delta^3 f$
x_0	f_0			
		$f(x_0, x_1)$		
x_1	f_1		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	f_2		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$
x_3	f_3		$f(x_2, x_3, x_4)$	
		$f(x_3, x_4)$		
x_4	f_4			
$\vdots$	$\vdots$			
x_n	f_n			

where

$$f(x_0, x_1) = \frac{f_1 - f_0}{x_1 - x_0},$$

In general

$$f(x_i x_j) = \frac{f_j - f_i}{x_j - x_i} \quad , \quad i \neq j \;,\; j = i + 1$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_0, x_1, x_2, \dots, x_n) = \frac{f(x_1, x_2, x_3, \dots, x_n) - f(x_0, x_1, x_2, \cdots, x_{n-1})}{x_n - x_0}$$

DIFFERENCE TABLES (Equal Intervals):

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = h$$

x_i	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\cdots$	$\Delta^n f$
x_0	f_0					
		Δf_0				
x_1	f_1		$\Delta^2 f_0$			
		Δf_1		$\Delta^3 f_0$		
x_2	f_2		$\Delta^2 f_1$			
		Δf_2		$\Delta^3 f_1$		
x_3	f_3		$\Delta^2 f_2$			
		Δf_3				
x_4	f_4					
$\vdots$	$\vdots$					
x_n	f_n					

where

$$\Delta f_0 = f_1 - f_0, \qquad \text{in general} \qquad \Delta f_i = f_{i+1} - f_i$$
$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0, \qquad \text{in general} \qquad \Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$
$$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0, \qquad \text{in general} \qquad \Delta^3 f_i = \Delta^2 f_{i+1} - \Delta^2 f_i$$

and so no

$$\Delta^n f_0 = \Delta^{n-1} f_1 - \Delta^{n-1} f_0, \qquad \text{in general} \qquad \Delta^n f_i = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i$$

FORMULE FOR INTERPOLATION OF EQUAL INTERVAL DATA:

x_i	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\cdots$	$\Delta^n f$
x_{-4}	f_{-4}						
		Δf_{-4}					
x_{-3}	f_{-3}		$\Delta^2 f_{-4}$				
		Δf_{-3}		$\Delta^3 f_{-4}$			
x_{-2}	f_{-2}		$\Delta^2 f_{-3}$		$\Delta^4 f_{-4}$		
		Δf_{-2}		$\Delta^3 f_{-3}$			
x_{-1}	f_{-1}		$\Delta^2 f_{-2}$		$\Delta^4 f_{-3}$		
		Δf_{-1}		$\Delta^3 f_{-2}$			
x_0	f_0		$\Delta^2 f_{-1}$		$\Delta^4 f_{-2}$		
		Δf_0		$\Delta^3 f_{-1}$			
x_1	f_1		$\Delta^2 f_0$		$\Delta^4 f_{-1}$		
		Δf_1		$\Delta^3 f_0$			
x_2	f_2		$\Delta^2 f_1$				
		Δf_2					
x_3	f_3						
$\vdots$	$\vdots$						
x_n	f_n						

(i) Newton's Forward $(x_0 \ to \ x_n)$

$$P_n(x) = f_0 + \binom{S}{1}\Delta f_0 + \binom{S}{2}\Delta^2 f_0 + \binom{S}{3}\Delta^3 f_0 + \cdots + \binom{S}{n}\Delta^n f_0$$

(ii) Newton's Backward $(x_{-n} \ to \ x_0)$

$$P_n(x) = f_0 + \binom{S}{1}\Delta f_{-1} + \binom{S+1}{2}\Delta^2 f_{-2} + \cdots + \binom{S+n-1}{n}\Delta^n f_{-n}$$

or
$$P_n(x) = f_0 + \binom{S}{1}\nabla f_0 + \binom{S+1}{2}\nabla^2 f_0 + \cdots + \binom{S+n-1}{n}\nabla^n f_0$$

where $S = \dfrac{x - x_0}{h}$; h = width of the interval

The forward difference formula is suitable if you want to interpolate near the top of q set of tabulated values.

For example if you want to calculate the value of

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