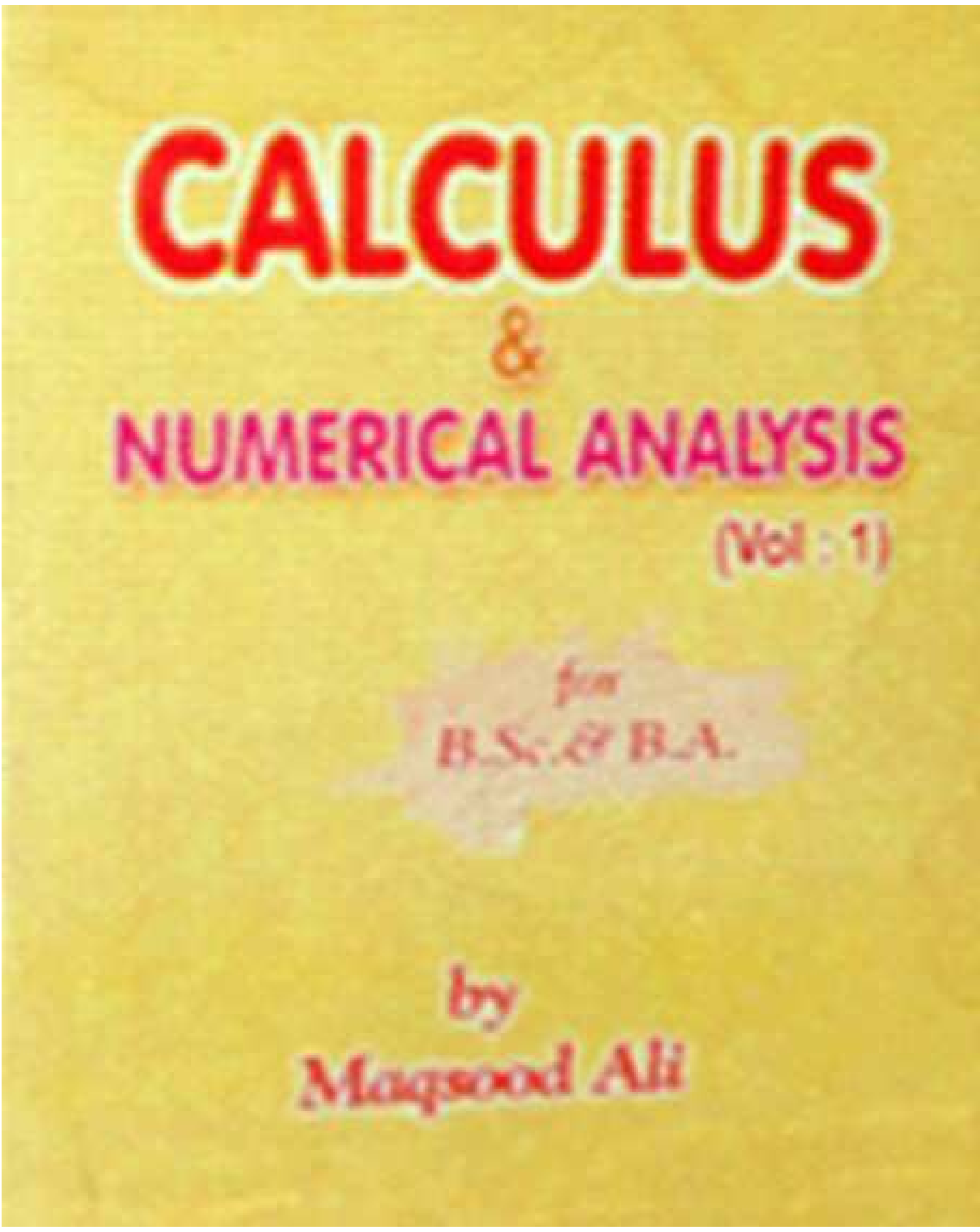
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
Maqsood Ali

ITERATIVE METHODS

We use iterative methods to find a root of a equation $f(x) = 0$ either by computer or by hand some time. We will study five different iterative methods in this section.
To understand the iterative methods first we solve a quadratic equation by quadratic formula, by the graph and then by five different iterative methods.
The quadratic equation is

$$x^2 - 2.5x - 4 = 0$$

(a) Solution by quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2.5 \pm \sqrt{6.25 + 16}}{2}$$

$$x = 3.608, \qquad -1.108$$

(b) Solution by graph:

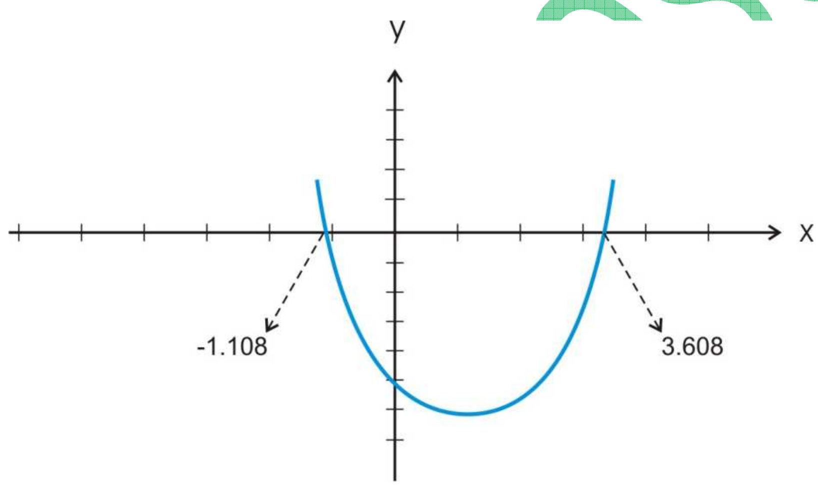

Figure 1

The roots of the equation are the points where the curve $y = f(x)$ cuts the x-axis.

METHOD OF HALVING THE INTERVAL OR METHOD OF BISECTION:

Let $$f(x) = x^2 - 2.5x - 4$$

We find the value of x where $f(x)$ changes the sign or $f(x) = 0$

$$f(0) = -4\,,\quad f(1) = -5.5,\quad f(2) = -5,\quad f(3) = -2.5,\quad f(4) = +2$$

The opposite signs of $f(3)$ and $f(4)$ show that at least one root belong to the interval $(3, 4)$.

The mid-point of this interval is 3.5, thus we divide the interval $(3, 4)$ into two intervals $(3,3.5)$ and $(3.5,4)$. We calculate the value of f at $x = 3.5$

$$f(3.5) = -0.5$$

The value $f(3)$ and $f(3.5)$ have same sign but $f(3.5)$ and $f(4)$ have opposite signs. Hence a root lies between $(3.5,4)$. The mid-point of the interval $(3.5,4)$ is 3.75, Now we have two intervals $(3.5,3.75)$ and $(3.75,4)$.

$$f(3.75) = +0.6875$$

Since value of $f(3.75)$ and $f(3.5)$ have different sign, hence a root lies between $(3.5,3.75)$.

Continuing this process we get smaller and smaller interval with in which root lie. This process is shown below.

$$f(x) = x^2 - 2.5x - 4$$

$f(3) = -2.5 < 0$	$f(3.5) = -0.5$	$f(4) = +2 > 0$
$f(3.5) < 0$	$f(3.75) = +\,0.6875$	$f(4) > 0$
$f(3.5) < 0$	$f(3.625) = +0.078125$	$f(3.75) > 0$
$f(3.5) < 0$	$f(3.5625) = -0.371$	$f(3.625) > 0$
$f(3.5625) < 0$	$f(3.59375) = -0.069$	$f(3.625) > 0$
$f(3.59375) < 0$	$f(3.609375) = +$	$f(3.625) > 0$
$f(3.59375) < 0$	$f(3.6015625) = -0.0326$	$f(3.609375) > 0$
$f(3.6015625) < 0$		$f(3.609375) > 0$

A root lies between (3.6015625, 3.609375)

Hence the one root correct to two decimal places is 3.60.

$$x = 3.60$$

Example 8:

$f(1) = -2 < 0$	$f(1.5) = 30.171875$	$f(2) = 23 > 0$
$f(1) < 0$	$f(1.25) = 1.109$	$f(1.5) > 0$
$f(1) < 0$	$f(1.125) = -0.666$	$f(1.25) > 0$
$f(1.125) < 0$	$f(1.1875) = 0.164$	$f(1.25) > 0$
$f(1.125) < 0$	$f(1.15625) = -0.265$	$f(1.1875) > 0$
$f(1.15625) < 0$	$f(1.171875) = -0.05$	$f(3.625) > 0$
$f(1.711875) < 0$	$f(1.1796875) = -0.054$	$f(1.1875) > 0$
$f(1.171875) < 0$		$f(1.1796875) > 0$

One root of the given equation $x = 1.17$ (correct to two decimal places)?

Important points:

(1) Bisection method is very slow to converge.
(2) It is very simplest iterative method.
(3) It is guaranteed to converge to the root, if the starting values x_1 and x_2 be the end points of the interval in which the root lie.
If a root lies in the interval (x_1, x_2) then $f(x_1).f(x_2) < 0$
because a root must lie between the x −values where the function changes the sign.
(4) The roots of the equation are the points where the graph of $y = f(x)$ cuts the x-axis.
The starting values can also get by making a rough graph, by trial calculation.

EXERCISE D-4

Find a root (correct to 3 decimal) of the following by bisection method with given starting values.

(1) $e^x sin\sqrt{1+x^2} - 3\, In\, x^2 + 2 = 0$, $(-1, -2)$
(2) $In\,(x+1) - \sqrt{x-1} = 0$, $(1,3)$
(3) $x^2 \cos^2 x + 5\, e^x - 7 = 0$, $(-\pi/4\ , \pi/4)$.

Find the greatest negative root of the following equation by bisection method.

(4)
(5) $2x^5 + 3x^4 + 8 = 0$

Using bisection method find the smallest positive real root of the following equations.

(6) $x^2 - 7x + 11 = 0$
(7) $e^x - 3x = 0$
(8) $x^3 In\sqrt{1+x} - 3x^2 - 1 = 0$
(9) $2x^3 - 3x^2 + x - 5 = 0$
(10) $2x^3 - 9x^2 + 7x - 3 = 0$

(II) THE METHOD OF FALSE POSITION:

This procedure begins by locating two points x_0 and x_1 where the function has opposite signs. The two points $f(x_0)$ and $f(x_1)$ are connected by a straight line and we find where it cuts the x-axis. If it cut the x-axis at x_2 then we find $f(x_2)$. If $f(x_2)$ and $f(x_0)$ are of opposite signs then we replace x_1 by x_2 and draw a straight line connecting $f(x_2)$ to $f(x_0)$ to find the new intersection point.

If $f(x_2)$ and $f(x_0)$ are of some sign then x_0 is replaced by x_2 and proceed as before. In both cases the new interval is smaller than the initial interval.

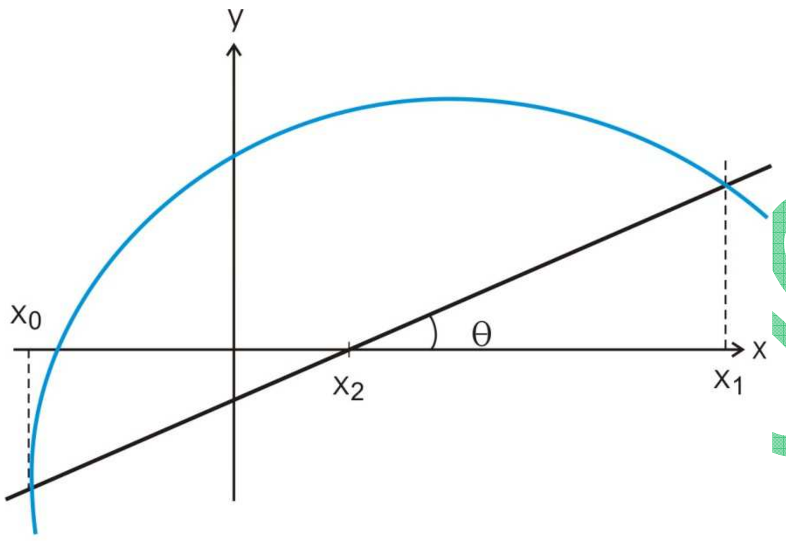

Figure 2

According to the above figure

$$\tan\theta = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad \rightarrow (1)$$

and

$$\tan\theta = \frac{f(x_1)}{x_1 - x_2} \qquad \rightarrow (2)$$

By equation (1) and (2)

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1)}{x_1 - x_2}$$

$$x_1 - x_2 = f(x_1)\frac{(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

In general

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Example 9:

Find a real root of the following equation:

$$x^3 - x^2 - 3x - 5 = 0$$

Solution:

Let $f(x) = x^3 - x^2 - 3x - 5 = 0$

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

n	x_{n-1}	xn	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	2	3	−7	4	2.636363	−1.535695
2	2.636363	3	−1.535695	4	2.737242	−0.195451
3	2.737242	3	−0.195451	4	2.749483	−0.022958
4	2.749483	3	−0.022958	4	2.750912	−0.002679
5	2.750912	3	−0.002679	4	2.751078	−0.000322
6	2.751078	3	−0.000322	4	2.751098	−0.000037
7	2.751098	3	−0.000037	4	2.751100	−0.000009
8	2.751100	3	−0.000009	4	2.751100	

The real root is 2.75110 correct to 5 decimals.

IMPORTANT POINTS:

(1) False position method is slow to converge.
(2) It is guaranteed to converge to the root.
(3) The procedure which we adopt in false position method is one sided, which puts effect on the speed.In modified false position method divides the ordinate by 2 $\{as \frac{f(x)}{2}\}$ at the third iteration.

Modified Form:

n	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	2	3	−7	4	2.636363	−1.535695
2	2.636363	3	−1.535695	4	2.737242	−0.195451
3	2.737242	3	−0.195451	2*	2.760634	−0.136065
4	2.737242	2.760634	−0.195451	0.13605	2.751033	−0.000961
5	2.751033	2.760634	−0.000961	0.13605	2.751100	−0.000009
6	2.51100	2.760634	−0.000009	0.068032	2.751101	

The real root is 2.75110 correct to 5 decimals.

*divide $f(x_n)$ by 2

EXERCISE D-5

Find a real root (correct to 3 decimals) of the following by False position method.

(1) $2x^3 - 6x^2 + 7x - 5 = 0$ (2) $3x^3 + 14x^2 + 20x + 16 = 0$

Find the greatest negative real root (correct to 3 decimals) of the following by false position method.

(3) $f(x) = x^3 + 15x^2 + 71x + 95 = 0$ (4) $2x^3 + 17x^2 + 19x - 5 = 0$

Find the smallest positive real root (correct to 3 decimals) by false position method.

(5) $In\, x - e^{1/x} = 0$ (6) $x^4 - 3x^3 - 30 = 0$

(7) $x^5 - e^x - 20 = 0$ (8) $x^4 + Inx^2 - 9 = 0$

(9) $x^6 - e^{2x} + Inx - 300 = 0$ (10) $x^3 + x^2 - 14x - 10 = 0$

Find a root (correct to 3 decimals) in the given interval by false position method.

(11) $\cos x + 3x^2 - e^x - 2 = 0$, $(0, \pi/2)$

(12) $x^4 \sin x - e^x + \frac{3}{2} = 0$, $(0, \pi/3)$

(13) $x^3 Inx - x^2\sqrt{x} - 6 = 0$, $(2,3)$

(14) $x\, e^x Inx - 20\sqrt{x} - 20 = 0$, $(2,3)$

(iii) NEWTON'S METHOD:

To find the roots of the equation $f(x) = 0$,we use a method which is called Newton's method or Newton Raphson method.

In this process f is assume to have a continuous derivative f'.

Geometrical Explanation:

Consider a curve $f(x)$ as shown in the figure. A tangent to the curve is drawn at an approximate value x_0 obtained from the graph and the point where the tangent intersect x-axis is taken as the next approximation x_1.

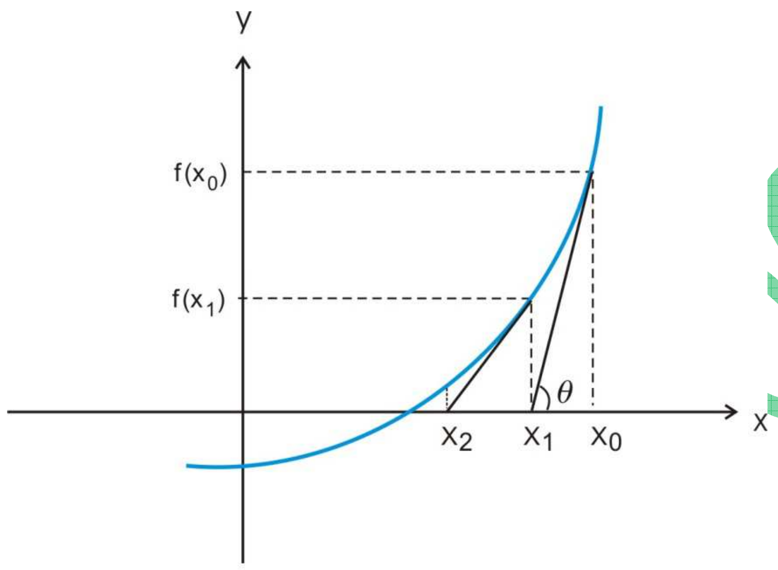

Figure 3

By the graph

$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

or
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In second step a tangent to the curve f is drawn at x_1, and the point where tangent intersect x-axis is taken as the next approximation value x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The same procedure is repeated till $f(x_n)$ becomes nearly zero, in turn implies that x_n and x_{n+1} are almost equal.

Formula:

$$x_{n+1} = x_n - \frac{f(x_1)}{f'(x_1)} \qquad , n = 0,1,2,3, \cdots$$

Examples 10:

Use Newton's method to find a root of the following equations.

(1) $x^2 - 2.5x - 4 = 0$

(2) $3x^3 + 4x^2 - 8x - 1 = 0$

(3) $3x^3 + 4x^2 - 8x - 1 = 0 \quad ; \quad x_0 = 4$

(4) $3x^3 + 4x^2 - 8x - 1 = 0 \quad ; \quad x_0 = 8$

(5) $3x^3 + 4x^2 - 8x - 1 = 0 \quad ; \quad x_0 = 0$

(6) $x^3 - 4x + 1 = 0 \quad ; \quad x_0 \in (1, 3)$

Solution:

(1) $x^2 - 2.5x - 4 = 0$

Let
$$f(x) = x^2 - 2.5x - 4$$
$$f'(x) = 2x - 2.5$$
or
$$f(x_n) = x_n^2 - 2.5x_n - 4$$
$$f'(x_n) = 2x_n - 2.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_2 = 3.6111 - \frac{0.0123}{4.7222} = 3.6085$$

Substituting $n = 2$ in equation (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 3.6085 - \frac{0.00002}{4.717} = 3.6085$$

Since $x_2 \cong x_3$ ($x_2 = x_3$ upto four decimal places).
Hence $x = 3.6085$ is a root of the given equation.
Following table shows the results of using Newton's method starting with $x_0 = 3.5$.

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	x_{n+1}
0	3.5	−0.5	4.5	−0.1111	3.6111
1	3.6111	0.0123	4.7222	0.0026	3.6085
2	3.6085	0.00002	4.717	0.0000	3.6085
3	3.6085				

(2) $3x^3 + 4x^2 - 8x - 1 = 0$

Let $f(x)3x^3 + 4x^2 - 8x - 1$

$f'(x) = 9x^2 + 8x - 8$

$f(-1) = 8$, $f(0) = -1$, $f(1) = -2$, $f(2) = 23$

$f(-1) > 0$, $f(0) < 0$ *and* $f(1) < 0$, $f(2) > 0$

Show that a root lies in $(-1,0)$ and another in $(1,2)$.
We find only one root in $(1,2)$
Since $f(1) = -2 < 0$ and $f(2) = 23 > 0$ and $f(x)$ is continuous thus $f(x)$ has a root in the interval $(1,2)$.
Suppose we guess that $x = 1.5$ is approximately the root.
Approximation value is $x_0 = 1.5$
According to Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \rightarrow (1)$$

Substituting $n = 0$ in equation (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \frac{6.125}{24.25} = 1.2474$$

Substituting $n = 1$ in equation (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_2 = 1.2474 - \frac{1.0677}{15.9832} = 1.1306$$

Substituting $n = 2$ in equation (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.1806 - \frac{0.0671}{13.9891} = 1.1758$$

Substituting $n = 3$ in equation (1), we get

$$x_4 = 1.1758 - \frac{0.00027}{13.8489} = 1.17578$$

Since $x_3 = x_4 = 1.175$

Hence one root of the given equation is $x = 1.175$

Table shows the results of using Newton's method starting with$x_0 = 1.5$

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	x_{n+1}
0	1.5	6.125	24.25	0.2525	1.2474
1	1.2474	1.0677	15.9832	0.0668	1.1806
2	1.1806	0.0671	13.9891	0.00479	1.1758
3	1.11758	0.00027	13.8489	0.00001	1.17578
4	1.17578				

(3) $f(x) = 3x^3 + 4x^2 - 8x - 1$; $x_0 = 4$

$f'(x) = 9x^2 + 8x - 8$

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	x_{n+1}
0	1.5	6.125	24.25	0.2525	1.2474
1	1.2474	1.0677	15.9832	0.0668	1.1806
2	1.1806	0.0671	13.9891	0.00479	1.1758
3	1.11758	0.00027	13.8489	0.00001	1.17578
4	1.17578				

$x_6 = x_7 = 1.1757$

Hence one root of the given equation is $x = 1.1757$

(4) $3x^3 + 4x^2 - 8x - 1 = 0 \quad , \quad x_0 = 8$

Let $f(x) = 3x^3 + 4x^2 - 8x - 1$

$f'(x) = 9x^2 + 8x - 8$

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	x_{n+1}
0	8	1727	632	2.7326	5.2674
1	5.2674	506.28	283.84	1.7837	3.4837
2	3.4837	156.51	129.09	1.1349	2.3487
3	2.3487	41.145	60.437	0.6808	1.6679
4	1.6679	10.704	30.38	0.3532	1.3155
5	1.3155	2.2277	18.099	0.1231	1.1924
6	1.1924	0.2341	14.335	0.0163	1.1760
7	1.1760	0.0030	13.854	0.0002	1.1757
8	1.1757				

$x_7 = x_8 = 1.17,\ x = 1.17$ (correct to 2 decimals places) is a root of the given equations.

(5) $3x^3 + 4x^2 - 8x - 1 = 0,\ \ x_0 = 0$

Let $f(x)3x^3 + 4x^2 - 8x - 1$

$f'(x) = 9x^2 + 8x - 8$

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	x_{n+1}
0	0	-1	-8	0.125	-0.125
1	-0.125	0.0566	-8.8593	-0.0658	-0.1186
2	-0.1186	0.00006	-8.8222	-0.0000	-0.11859
3	-0.11859				

$x_2 = x_3 = -0.118$ (correct to 3 decimals places) is a root of the given equation.

(6) $x^3 - 4x + 1 = 0 \quad ; \quad x \in (1, 3)$

Let $f(x) = x^3 - 4x + 1$

$f'(x) = 3x^2 - 4$

Suppose we guess that $x = 2$ is approximately the root.

Hence $x_0 = 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	x_{n+1}
0	0	-1	-8	0.125	-0.125
1	-0.125	0.0566	-8.8593	-0.0658	-0.1186
2	-0.1186	0.00006	-8.8222	-0.0000	-0.11859
3	-0.11859				

Since $x_3 = x_4$

Hence one root of the given equation is $x = 1.86081$

IMPORTANT POINTS:

(1) This method has the fastest convergence.

(2) When the method is successful the number of accurate places of the estimate at each step is roughly twice the number of accurate places of the previous step. For example, suppose. an estimate of the root is accurate to three decimal places ; then the next estimate is usually accurate to about six decimal places.

(3) The method does not always work. In fact, the method may be successful for one guess x_0 and fail for another choice of x_0. See Exercise 20, 21.

(4) This method can be failed if an unfortunate choice of x_0 results is a zero value for $f'(x_n)$.

(5) It is quite sensitive as it may diverge if $f'(x)$ is near zero any time during the iterative cycle.

EXERCISE D-6

Using Newton's method find a root (correct to 6 decimals) of the following equation with starting values x_0.

(1) $-3x^3 - 30 = 0$, $x_0 = 3$

(2) $2x^3 - 3x^2 + x - 50 = 0$, $x_0 = 0$

(3) $x\, e^x\, In\, x - 20\sqrt{x} - 20 = 0$, $x_0 = 3$

(4) $x^4 + In\, x^2 - 9 = 0$, $x_0 = 1$

(5) $x^5 - e^x - 20 = 0$, $x_0 = 3$

(6) $e^x - 3x = 0$, $x_0 = 0$

(7) $x^3\, In\, x - x^2\sqrt{x} - 6 = 0$, $x_0 = 2$

(8) $2x^3 + 17x^2 + 19x - 5 = 0$, $x_0 = -1$

(9) $x^2 \cos^2 x + 5e^x - 7 = 0$, $x_0 = \pi/4$

(10) $2\sin x - e^{2x} + x^2 - 2 = 0$, $x_0 = -\pi/3$

Using Newton's method find the smallest positive real root (correct to 3 decimals) of the following equations:

(11) $e^x + 4x - 9 = 0$

(12) $\ln x + x^2 + 3x = 0$

(13) $\cos x - 5x + 2x^2 = 0$

(14) $x^3 - 2x^2 + x - 13 = 0$

Find the greatest negative root (correct to 6 decimals) of the following equations, using Newton's method.

(15) $\sin x + 5x + 5 = 0$

(16) $2x^3 + 5x^2 + 3x + 9 = 0$

(17) $e^x + 2x + 9 = 0$

Find a real root of the following equations correct to 6 decimals, by Newton's method.

(18) $2x^3 - 6x^2 + 7x - 5 = 0$

(19) $3x^3 + 14x^2 + 20x + 16 = 0$

Show that Newton's method is unsuccessful for the following function with starting values x_0 but successful for another given starting value x_0.

(20) $f(x) = x^3 + 2x^2 - 7x + 5$, $x_0 = 1$, $x_0 = -4$

(21) $f(x) = x^3 - 12x + 24$, $x_0 = 0$, $x_0 = -5$

Show that the Newton's method is not successful to find a root of $f(x) = x^{1/3}$ (which is zero) with starting values $x_0 = 1$.

SECANT METHOD:

Secant method is almost same as Newton's method. By substituting the value of $f'(x_n)$ in Newton's method we obtain secant method.

Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

By putting $f'(x_n) = \frac{f(x_n) - f'(x_{n-1})}{x_n - x_{n-1}}$, we have

$$x_{n-1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Secant method is used instead of Newton's method where $f'(x)$ is long and much computer time is needed to evaluate it. A serious disadvantage of Newton's method is need to calculate $f'(x)$ in each iteration.

Geometrical Representation:

Consider a curve $f(x)$ as shown in the figure.

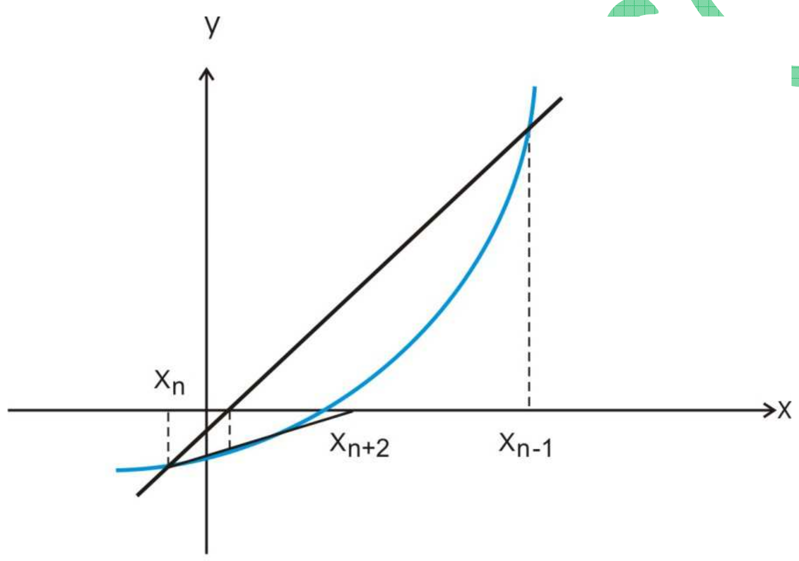

Figure 4

A secant is drawn connecting $f(x_{n-1})$ and $f(x_n)$ .The secant intersect x-axis at the point x_{n+1}.Another secant is drawn through $f(x_n)$ and $f(x_{n+1})$ which intersect x-axis at x_{n+2}. The same process is repeated till $f(x_n)$ becomes nearly zero.

Example 11:

Use secant method to find a root of the following equation.

$$x^2 - 2.5x - 4 = 0$$

Solution:

Let $$f(x) = x^2 - 2.5x - 4 = 0$$

$f(3) = -2.5 < 0, f(4) = +2 > 0$ shows that $f(x)$ has a root in the interval $(3, 4)$

$\Rightarrow \quad x_0 = 3$ and $x_1 = 4$

$f(x_0) = f(3) = -2.5$ and $f(x_1) = f(4) = 2$

According to secant method

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \qquad n = 1, 2, 3, \cdots \rightarrow (1)$$

Substituting $n = 1$ in equation (1) we get

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$= 4 - (2)\frac{4 - 3}{2 - (-2.5)} = 3.556$$

Substituting $n = 2$ in equation (1) we get

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

$$= 3.556 - (0.24486)\frac{3.556 - 4}{0.2248 - 2} = 3.4939$$

Substituting $n = 3$ in equation (1) we get

$$x_4 = x_3 - f(x_3)\frac{x_3 - x_2}{f(x_3) - f(x_2)}$$

$$= 3.4939 - (-0.5274)\frac{3.4939 - 3.556}{-0.5274 - 0.24486} = 3.53631$$

Substituting $n = 4$ in equation (1) we get

$$x_5 = x_4 - f(x_4)\frac{x_4 - x_3}{f(x_4) - f(x_3)}$$

$$= 3.2363 -\!\!- 0.3353\frac{3.5363 - 3.4939}{-0.3353 - (-0.5274)}$$

$$= 3.6103$$

Substituting $n = 4$ in equation (1) we get

$$x_6 = x_5 - f(x_5)\frac{x_5 - x_4}{f(x_5) - f(x_4)}$$

$$= 3.6103 - (-0.0085)\frac{3.6103 - 3.5363}{0.0085 - (-0.3353)} = 3.6084$$

Substituting $n = 5$ in equation (1) we get

$$x_7 = x_6 - f(x_6)\frac{x_6 - x_5}{f(x_6) - f(x_5)}$$

$$= 3.6084 - (-0.0004)\frac{3.6084 - 3.6103}{-0.0004 - 0.0085} = 3.6084$$

Since $x_6 = x_7 = 3.6084$
3.6084 is a root of the given equation.
Following table shows the results of using secant method starting with(3,4).

n	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}
1	3	4	−2.5	2	3.556
2	4	3.556	2	0.24486	3.4939
3	3.556	3.4939	0.24486	−0.5274	3.5363
4	3.4939	3.5363	−0.5274	−0.3353	3.6103
5	3.5363	3.6103	−0.3353	0.0085	3.6084
6	3.6103	3.6084	0.0085	−0.0004	3.6084

IMPORTANT POINTS:

(1) This method is not guaranteed to converge.
(2) It however, converges faster than the "False position method".
(3) It is an improved form of false position method. In this method we choose the last two values (nearest to the root) for interpolation, instead of two values at which the function has opposite sign as we do in false position method.

EXERCISE D-7

Using method find a root (correct to 3 decimals) of the following equations in the indicated interval.

(1) $x^4 - 3x^2 - 30 = 0$, (3,4)
(2) $x^3 Inx - x^2\sqrt{x} - 6 = 0$, (2,3)
(3) $xe^x Inx - 20\sqrt{x} - 20 = 0$, (2,3)
(4) $x^4 + Inx^2 - 9 = 0$, (1,2)

Using secant method find the smallest positive root of the following equations.

(5) $2x^3 - 3x^2 + x - 50 = 0$
(6) $x^5 - e^x - 20 = 0$

Using secant method find the greatest negative root (correct to 3 decimals) of the following equations.

(7) $3x^3 + 14x^2 + 20x + 16 = 0$
(8) $2x^3 + 17x^2 + 19x - 5 = 0$

FIXED POIND METHOD
OR
METHOD OF SUCCESSIVE APPOXIMATION:

In this method the equation $f(x) = 0$ is expressed as $x = g(x)$.
We can write it as

$$x_{n+1} = (g(x_n) \quad , \quad n = 0,1,2,3,\cdots$$

This method is start with a guess value x_0.
By substituting $x_n = x_0$, we get

$$x_1 = g(x_0)$$

By taking $g(x_0)$ as the next guess, we get $x_2 = g(x_1)$
The same procedure is repeated till two successive values of x are close enough.
Thus we can write a formula for this method.

$$x_1 = g(x_0),\ x_2 = g(x_1),\ x_3 = g(x_2), \cdots, x_n = g(x_{n-1})$$

This process will converge to the root if $|g'(x_n)| < 1$

Example 12:

Using fixed point method find the roots of the following equation.

$$x^2 - 2.5x - 4 = 0 \quad , \ x_0 = 3$$

Solution:

The given equation is

$$f(x) = x^2 - 2.5x - 4 = 0 \quad , \ x_0 = 3$$

Rearrange it into an equivalent expression of the form

$$x = g(x)$$

First Rearrangement:

$$x = \sqrt{2.5x + 4}$$
$$g(x) = \sqrt{2.5x + 4}$$
$$g'(x) = 1.25/\sqrt{2.5x + 4}$$
$$|g'(x)| = |1.25/\sqrt{2.5x + 4}\ |$$

$|g'(x)| < 1$ for all $x \in (-0.975, \infty)$, except $x \in (-\infty, -0.975)$
So x will converge and a root belongs to the interval $(-0.975, \infty)$.

$$x_{n+1} = \sqrt{2.5x_n + 4} \quad , \quad n = 0,1,2,3,\cdots$$
$$x_1 = \sqrt{2.5x_0 + 4} = \sqrt{2.5 \times 3 + 4} = 3.3911$$
$$x_2 = \sqrt{2.5x_1 + 4} = \sqrt{2.5 \times 3.3911 + 4} = 3.5324$$
$$x_3 = \sqrt{2.5x_1 + 4} = \sqrt{2.5 \times 3.5324 + 4} = 3.5820$$
$$x_4 = \sqrt{2.5x_3 + 4} = \sqrt{2.5 \times 3.5820 + 4} = 3.5993$$
$$x_5 = \sqrt{2.5x_4 + 4} = \sqrt{2.5 \times 3.5993 + 4} = 3.6053$$
$$x_6 = \sqrt{2.5x_5 + 4} = \sqrt{2.5 \times 3.6053 + 4} = 3.6073$$
$$x_5 = x_6 = 3.60$$

$x = 3.60$ (correct to 2 decimals) is a root of the given equation.

Second Rearrangement:

$$x = 4/(x - 2.5)$$

Here $$g(x) = 4/(x - 2.5)$$

$$g'(x) = \frac{-4}{(x-2.5)^2}$$

$$|g'(x)| = \left|\frac{4}{(x-2.5)^2}\right|$$

$|g'(x)| < 1$ for all $x \in$ R, except $0.5 < x < 4.25$

So x will converge, and the root belong to $(-\infty, 0.5) \cap (4.5, \infty)$

$$x_{n+1} = 4/(x_n - 2.5)$$

$$x_1 = 4/(x_0 - 2.5) = 4/(3 - 2.5) = 8$$

$$x_2 = 4/(x_1 - 2.5) = 4/(8 - 2.5) = 0.7273$$

$$x_3 = 4/(x_2 - 2.5) = 4/(0.7273 - 2.5) = -2.2564$$

$x_4 = -0.8409$ $x_5 = -1.1973$

$x_6 = -1.0819$ $x_7 = -1.1167$

$x_8 = -1.110598$ $x_9 = -1.1093$

$x_{10} = -1.1085$ $x_{11} = -1.1085$

$x_{12} = -1.10849$

$x = -1.108$ is a root of the given equation.

Third Rearrangement:

$$x = (x^2 - 4)/2.5$$

Here $$g(x) = (x^2 - 4)/2.5$$

$$g'(x) = 0.8x$$

$$|g'(x)| = |0.8x|$$

$|g'(x)| > 1$ for all $x \in R$, except $-1.25 < x < 1.25$

So x will not converge, as given below

$$x_{n+1} = (x_n^2 - 4)/2.5$$

$$x_1 = (x_0^2 - 4)/2.5 = (3^2 - 4)/2.5 = 2$$

$x_2 = 0$ $x_3 = -1.6$

$x_4 = -0.576$ $x_5 = -1.4672$

$x_6 = -0.7389$ $x_7 = -1.3816$

$x_8 = -0.8364$ $x_9 = -1.3202$

$x_{10} = -0.9028$ $x_{11} = -1.2739$

The root does not exist.

Example 13:

Using fixed point method finds a root of the following equation.

$$3x^3 + 4x^2 - 8x - 1 = 0$$

Solution:

$$x = g(x)$$

We can find a root by the arrangement

$$x = \left(\frac{-4x^2 + 8x + 1}{3}\right)^{1/3}$$

or $x_{n+1} = \left(\frac{4x_n^2 + 8x_n + 1}{3}\right)^{1/3}$, $n = 0,1,2,3,\cdots$

$$x_1 = \left(\frac{-4x_0^2 + 8x_0 + 1}{3}\right)^{1/3} = \left(\frac{-4 \times 1^2 + 8 \times 1 + 1}{3}\right)^{1/3} = 1.1856$$

$$x_2 = \left(\frac{-4x_1^2 + 8x_1 + 1}{3}\right)^{1/3} = \left(\frac{-4(1.1856)^2 + 8(1.1856) + 1}{3}\right)^{1/3} = 1.1746$$

$$x_3 = \left(\frac{-4x_2^2 + 8x_2 + 1}{3}\right)^{1/3} = \left(\frac{-4(1.1746)^2 + 8(1.1746) + 1}{3}\right)^{1/3} = 1.1759$$

$$x_4 = \left(\frac{-4x_3^2 + 8x_3 + 1}{3}\right)^{1/3} = \left(\frac{-4(1.1756)^2 + 8(1.1759) + 1}{3}\right)^{1/3} = 1.1757$$

$x_3 = x_4 = 1.175$

Hence one root of the given equation is $x = 1.175$

IMPORTANT POINTS:

(1) It is not guaranteed to convergence.

(2) It is suitable in some special cases.

(3) It is easy to program.

EXERCISE D-8

Find the positive root (correct to 6 decimals) of the following equations by Fixed Point Method.

(1) $x^2 + 2x - 32 = 0$

(2) $x - e^x + 3 = 0$

(3) $3x^3 + 14x^2 + 20x + 16 = 0$, $x_0 = -3$

(4) $x^6 - e^{2x} + Inx - 300 = 0$, $x_0 = 2$

(5) $f(x) = e^{2x} - 8x^2 = 0$ has three roots. Write down the three different arrangements and then find a root by an arrangement

$$x = \frac{e^x}{\sqrt{8}}$$

with starting value $x_0 = 0.743$ (correct to 3 decimals)

AUTHOR

M. MAQSOOD ALI

ASSISTANT PROFESSOR OF MATHEMATICS

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M. MAQSOOD ALI

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