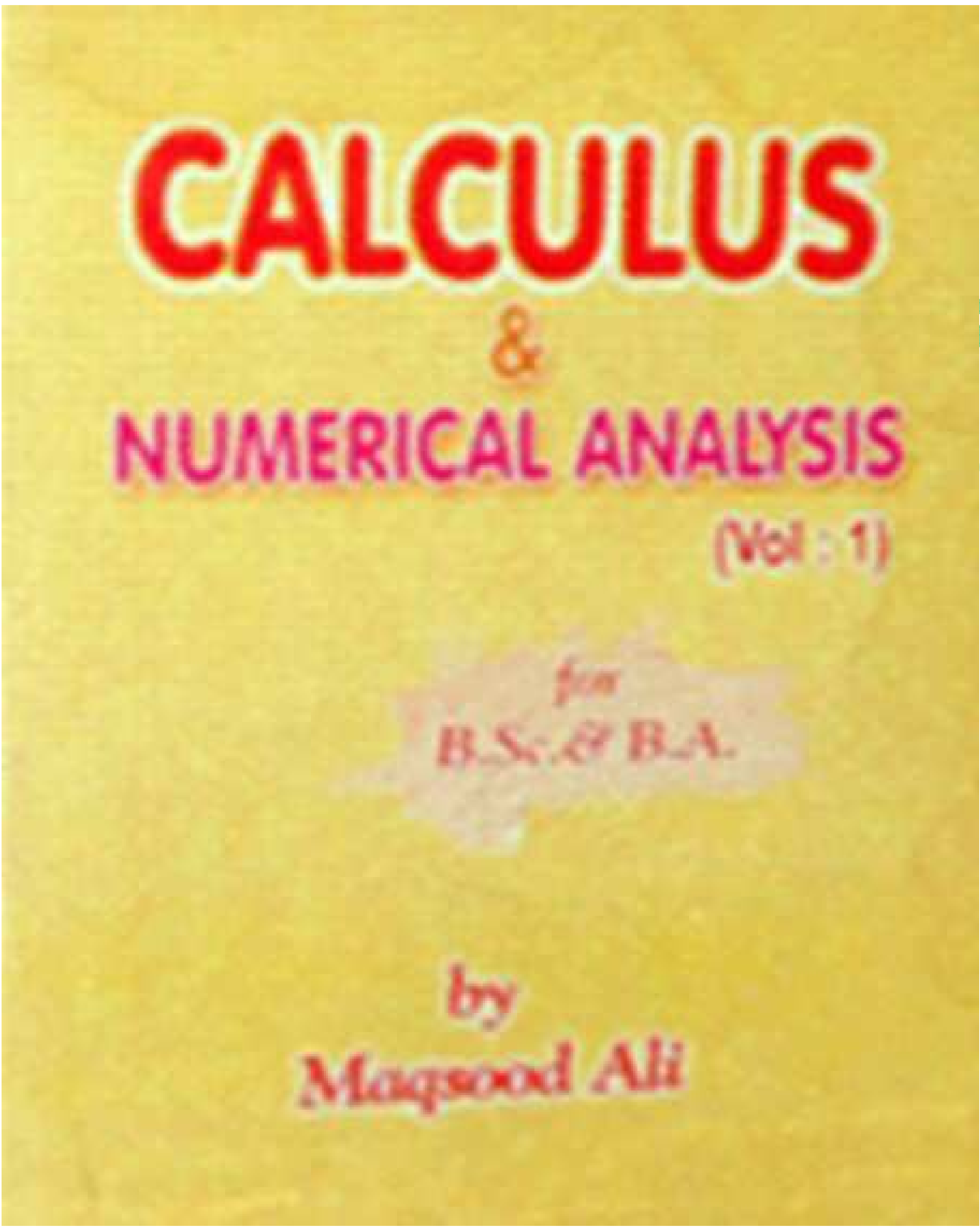
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
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ALGEBRAICAL SOLUTION OF EQUATIONS

ALGEBRA is a branch of mathematics. When we study Higher Algebra we see that a quadratic, cubic and biquadratic equations can be solved by Quadratic formula, Cardan's method, Ferrari's method respectively. But the general algebraical solution of equations of a degree higher than the fourth has not been obtained.
The study of "NUMERICAL ANAYTSIS" tell us how can we find the roots of a polynomial equation of degree n. A cubic, quartic or a polynomial equation of degree higher than fourth may be solved by a very famous method which is called Q-D method.
First we discuss the algebrical solution of cubic and biquadratic equations.

Cardan's Method (Solution of a Cubic Equation):

Consider the following cubic equation:-

$$x^3 + Ax^2 + Bx + C = 0 \qquad \rightarrow (1)$$

remove the term involving x^2

$$y^3 + by + C = 0 \qquad \rightarrow (2)$$

To remove the term involving x^2:

To remove the second term from the given equation

$$Px^3 + Ax^2 + Bx + C = 0$$

Let α, β and γ are the roots of equation, so that the sum of the roots is equal to $-A/P$.

$$\alpha + \beta + \gamma = -A/P$$

If we increase each of the roots by A/3P so we get a new equation by substituting $x = y - A/3P$ in the given equation, whose sum of the roots will be equal to $-A/P$, $+A/P$ or the co-efficient of second term will be zero. The new equation can be written as

$$Py^3 + by + c = 0$$

To solve the equation (2)

Put $y = s + t$, then

$$y^3 = (s+t)^3 = s^3 + t^3 + 3st\,(s+t) = s^3 + t^3 + 3sty \qquad \rightarrow (3)$$

So the equation(2) can be written as

$$s^3 + t^3 + 3sty + by + c = 0$$

$$s^3 + t^3 + (3st + b)y + c = 0 \qquad \rightarrow (4)$$

Put $\quad 3st + b = 0 \qquad \rightarrow (5)$

then $\quad s^3 + t^3 = -c \qquad \rightarrow (6)$

Equation (5) can be written as

$$st = -\frac{b}{3} \quad \Rightarrow \quad s^3t^3 = -b^3/27$$

Since $s^3 + t^3 = -c$ and $s^3t^3 = -b^3/27$, so that s^3 and t^3 are the roots of the equation

$$z^2 - (s^3 + t^3)\, z + (s^3t^3) = 0$$

$$z^2 + cz - \frac{b^3}{27} = 0 \qquad \rightarrow (7)$$

Solving this equation, we have

$$z = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}} \;, \quad -\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}$$

Since s^3 and t^3 are the roots of the equation(7) so that

$$s^3 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}} \;, \qquad t^3 = -\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}$$

The value of y is obtained from the equation

$$y = s + t$$

$$y = \left\{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}\right\}^{1/3} + \left\{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}\right\}^{1/3}$$

The above solution is generally known as Cardan's solution.
In this case we obtain nine values for$x$,
Because $s^3 + t^3 = -c$ and $s^3t^3 = -b^3/27$,
s^3t^3 would hold if $st = -\frac{b}{3}, \frac{-\omega b}{3}, \frac{-\omega^2 b}{3}$
We obtain three values of y substituting $st = -b/3$ in equation(3) (where $s^3 + t^3 = -c$)

$$y^3 + by + c = 0$$

others six values are obtain from the cubic equations
$y^3 + \omega by + c = 0$, $y^3 + \omega^2 by + c = 0$
{Note: these equations obtain by putting $st = -\omega b/3$ and $st = -\omega^2 b/3$ in equation(3), where $s^3 + t^3 = -c$}
As we know that equation (2) is a cubic equation , So it has only three cube roots.
Since $st = -b/3$, the cube roots are to be taken in pairs so that the product of each pair is rational.The $\omega^2 s, \omega t$ which fulfill the conditions are s, t or $\omega s, \omega^2 t$ or $\omega^2 s, \omega t$ where ω is the cube root of unity.

So that the roots of equation(2) are

$$y = s + t,\ \omega s + \omega^2 t,\ \omega^2 s + \omega t$$

The roots of equation (1) are

$$x = y - A/3p$$

Irreducible Case $[y = (m + in)^{1/3} + (m - in)^{1/3}]$:

If $s = (m + in)^{1/3}$ and $t = (m - in)^{1/3}$, then

Put $m = rcos\theta$ and $n = rsin\theta$,

where $r = \sqrt{m^2 + n^2}$ and $\theta = tan^{-1}\dfrac{n}{m}$ so that

$$(m + in)^{1/3} = \{r(cos\theta + i\,sin\theta)\}^{1/3}$$

By De Moivre's theorem the three values are

$$r^{\frac{1}{3}}(cos\,{}^{\theta}/_{3} + i\,sin\,{}^{\theta}/_{3})\ ,\ r^{1/3}(cos\frac{\theta + 2\pi}{3} + i\,sin\frac{\theta + 2\pi}{3})$$

and $r^{1/3}(cos\dfrac{\theta + 4\pi}{3} + i\sin\dfrac{\theta + 4\,\pi}{3})$

The three values of $(a - ib)^{1/3}$ are

$$r^{\frac{1}{3}}(cos\,{}^{\theta}/_{3} + i\,sin\,{}^{\theta}/_{3})\ ,\ r^{1/3}(cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3})$$

and $r^{1/3}(cos\dfrac{\theta + 4\pi}{3} + i\sin\dfrac{\theta + 4\,\pi}{3})$

Hence the roots are

$$2\,r^{1/3}\,cos\,{}^{\theta}/_{3}\ ,\ \ 2\,r^{1/3}\ \ cos\frac{\theta + 2\pi}{3}, 2\,r^{1/3}cos\frac{\theta + 4\pi}{3}$$

Example 2:

Solve $x^3 + 12x^2 + 66x + 117 = 0$ $\rightarrow (1)$

Solution:

To remove second term put $y - 4$ for x

$$(y - 4)^3 + 12(y - 4)^2 + 66(y - 4) + 117 = 0$$

$$y^3 + 18y - 19 = 0 \qquad \rightarrow (2)$$

Let $y = s + t$

$$s^3 + t^3 + (3st + b)\,y + c = 0$$

$$s^3 + t^3 + (3st + 18)\,y - 19 = 0$$

Put $3st + 18 = 0$ $\Rightarrow$ $s^3t^3 = -216$

then $s^3 + t^3 = 19$

s^3 and t^3 are the roots of the equation

$$z^2 - 19z - 216 = 0 \qquad \rightarrow (3)$$

$$z = -8\,, 27$$

Hence $s^3 = -8 \quad \Rightarrow \quad s = -2$

and $t^3 = 27 \quad \Rightarrow \quad t = 3$

The roots of equation (2) are

$y = s + t = -2 + 3 = 1$

$$y = \omega s + \omega^2 t = \frac{-1+\sqrt{-3}}{2}(-2) + \frac{-1-\sqrt{-3}}{2}(3) = \frac{-1-5\sqrt{-3}}{2}$$

$$y = \omega^2 s + \omega t = \frac{-1+5\sqrt{-3}}{2}$$

Since $x = y - 4$, so the roots of equation(1) are

$$x = -3\,, \quad \frac{-9-5\sqrt{-3}}{2}\,, \quad \frac{-9+5\sqrt{-3}}{2}$$

FERRARI'S METHOD (Biquadratic Equations):

We solve a Biquadratic Equation by Ferrar's method.

Consider the following biquadratic equation

$$x^4 + 2ax^3 + bx^2 + 2cx + d = 0$$

Adding both side $(mx+n)^2$

$$x^4 + 2ax^3 + bx^2 + 2cx + d + (mx+n)^2 = (mx+n)^2$$

$$x^4 + 2ax^3 + (m^2+b)x^2 + 2(mn+c)x + (n^2+d) = (mx+n)^2$$

Suppose that the left hand side is equal to$(x^2+ax+k)^2$

$$x^4 + 2ax^3 + (m^2+b)x^2 + 2(mn+c)x + (n^2+d) = (x^2+ax+k)^2$$

$$x^4 + 2ax^3 + (m^2+b)x^2 + 2(mn+c)x + (n^2+d) = x^4 + 2ax^3 + (2k+a^2)\,x^2 + 2akx + k^2$$

By equating the coefficients, we have

$$m^2 = 2k + a^2 - b\,, mn = ak - c\,, n^2 = k^2 - d$$

To eliminate m and n, we write

$$m^2n^2 = (mn)^2$$

$$(2k + a^2 - b)(k^2 - d) = (ak - c)^2$$

$$2k^3 - bk^2 + 2(ac - d)k + (bd - a^2d - c^2) = 0$$

This equation is satisfied for one real value of k. To find either real value of k is positive or negative, suppose that left hand side is equal to $f(k)$. Substituting $+\infty, 0, -\infty$for k, we have

$$f(+\infty) = \infty \ , \ f(0) = r \ , \ f(-\infty) = -\infty$$

If r is positive, then k is negative, if r is negative then k is positive.

We can obtain the value of m and n after finding the value of k. Since

$$(x^2 + ax + k)^2 = (mx + n)^2$$
$$x^2 + ax + k = \pm(mx + n)$$

We have two equations to find the values of x.

$$x^2 + (a - m)x + (k - n) = 0$$

and $$x^2 + (a + m)x + (k + n) = 0$$

When the Coefficient of x^4 is not equal to 1:

$$px^4 + 2ax^3 + bx^2 + 2cx + d = 0$$

In this case we suppose left hand side is equal to

$$p\left(x^2 + \frac{a}{p}x + k\right)^2.$$

Example 3:

Solve $x^4 + 2x^3 - 5x^2 + 6x - 3 = 0 \qquad \rightarrow (1)$

Solution:

Adding $(mx + n)^2$both side in equation (1)

$$x^4 + 2x^3 - 5x^2 + 6x - 3 + (mx + n)^2 = (mx + n)^2$$
$$x^4 + 2x^3 + (m^2 - 5)x^2 + (2mn + 6)x + (n^2 - 3) = (mx + n)^2 \qquad \rightarrow (2)$$

Suppose that the left hand side is equal to $(x^2 + x + k)^2$

$$x^4 + 2x^3 + (m^2 - 5)x^2 + (2mn + 6)x + (n^2 - 3) = (x^2 + x + k)^2$$
$$x^4 + 2x^3 + (m^2 - 5)x^2 + (2mn + 6)x + (n^2 - 3) = x^4 + 2x^3 + (1 + 2k)x^2 + 2kx + k^2$$

Equating the coefficients , we have

$m^2 - 5 = 1 + 2k \quad , 2mn + 6 = 2k \quad , \ n^2 - 3 = k^2$

$m^2 = 2k + 6 \qquad , mn = k - 3, \qquad , \ n^2 = k^2 + 3$

So that

$$m^2n^2 = (mn)^2$$

$(2k + 6) \ (k^2 + 3) = (k - 3)^2$

$2k^3 + 5k^2 + 12k + 9 = 0$

This equation is satisfied for $k = -1$

thus $\quad m^2 = 4 \qquad \Rightarrow \qquad m = 2$

and $\quad mn = k - 3 \qquad \Rightarrow \qquad n = -2$

we have supposed that

$$(x^2 + x + k)^2 = (mx + n)^2$$
$$(x^2 + x - 1)^2 = (2x - 2)^2$$

$$x^2 + x - 1 = \pm(2x - 2)$$

$$x^2 + x - 1 = (2x - 2) \quad \text{or} \quad x^2 + x - 1 = -(2x - 2)$$

$$x^2 - x + 1 = 0 \quad , \quad x^2 + 3x - 3 = 0$$

$$x = \frac{1 \pm \sqrt{-3}}{2} \quad , \quad x = \frac{-3 \pm \sqrt{-21}}{2}$$

METHOD OF DESCARTES:

Consider the following equation

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \qquad \rightarrow (1)$$

To remove the second term we divide successively by $\left(x + \frac{a_1}{4}\right)$, using synthetic divide method.

The transformed equation is

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

To find the roots of the above biquadratic equation suppose that

$$x^4 + ax^3 + bx^2 + cx + d = (x^2 + kx + m)\,(x^2 - kx + n)$$

$$x^4 + (m + n - k^2)\; x^2 + k(n - m)x + mn$$

by equating the conefficients, we have

$$m + n - k^2 = a \quad , \quad k(n - m) = b \quad , \quad mn = c$$

$$n + m = a + k^2 \quad , \quad n - m = b/k \quad , \quad mn = c$$

We obtain m and n by first two equations

so that $2m = a + k^2 - \frac{b}{k}$ and $2n = a + k^2 + \frac{b}{k}$

$$\Rightarrow \qquad m = \frac{k^3 + ak - b}{2k} \quad \text{and} \quad n = \frac{k^3 + ak + b}{2k}$$

Substituting these values in third equation, we have

$$(k^3 + ak - b)(k^3 + ak + b) = 4k^2c$$

$$k^6 + 2ak^4 + (a^2 - 4c)k^2 - b^2 = 0$$

$$(k^2)^3 + 2a(k^2)^2 + (a^2 - 4c)k^2 - b^2 = 0$$

Let $t = k^2$ then

$$t^3 + 2at^2 + (a^2 - 4c)t - b^2 = 0$$

It is a cubic equation which has a real positive root. Substituting the values of m, n and k in the following equation.

$$x^4 + ax^2 + bx + c = (x^2 + kx + m)(x^2 - kx + n) = 0$$

we find the value of x by the following equations.

$$x^2 + kx + m = 0 \quad , \quad x^2 - kx + n = 0$$

Note: To remove the second term of the following equation

$$b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5 = 0$$

Divide the equation by $\left(x + \frac{b_2}{4b_1}\right)$ using synthetic division method.

Example 4:

Solve $$x^4 - 12x^3 + 44x^2 - 56x + 20 = 0$$

Solution:-

To remove the second term, dividing by $(x - 3)$

3	1	−12	44	−56	20
		3	−27	5	−15
	1	−9	17	−5	5
		3	−18	−3	
	1	−6	−1	−8	
		3	−9		
	1	−3	−10		
		3			
	1	0			
	1				

Hence the result is

$$x^4 - 10x^2 - 8x + 5 = 0$$

Suppose that

$$x^4 - 10x^2 - 8x + 5 = (x^2 + kx + m)(x^2 - kx + n)$$
$$= x^4 + (m + n - k^2)\ x^2 + k(n - m)x + mn$$

By equating the coefficients, we have

$$m + n - k^2 = -10 \quad , \quad k(n - m) = -8 \quad , \quad mn = 5$$
$$n + m = -10 + k^2 \quad , \quad n - m = -\frac{8}{k} \quad , \quad mn = 5$$

By adding and subtracting first two equations, we get

$$n = \frac{k^3 - 10k - 8}{2k} \quad \text{and} \quad m = \frac{k^3 - 10k + 8}{2k}$$

Substituting these values in third equation, we have

$$\left(\frac{k^3 - 10k - 8}{2k}\right)\left(\frac{k^3 - 10k + 8}{2k}\right) = 5$$

or $$k^6 - 2ak^4 + 80k^2 - 64 = 0$$

$$(k^2)^3 - 20(k^2)^2 + 80k^2 - 64 = 0$$

Let $t = k^2$

$$t^3 - 20t^2 + 80t - 64 = 0$$

It is a cubic equation which has a root 4.

Thus $t = k^2 = 4 \quad \Rightarrow \quad k = \pm 2$

We consider only one value either $k = 2$ or $k = -2$

If $k = 2$ then $m = -1$ and $n = -5$

So that

$$x^4 - 10x^2 - 8x + 5 = (x^2 + 2x - 1)(x^2 - 2x - 5) = 0$$

$\Rightarrow \quad x^2 + 2x - 1 = 0 \quad , \quad x^2 - 2x - 5 = 0$

$\Rightarrow \quad x = -1 \pm \sqrt{2} \quad , \quad x = -1 \pm \sqrt{6}$

EXERCISE D-2

Solve the following equations by Cardan's method.

(1) $x^3 + 30x + 117 = 0$

(2) $x^3 - 12x - 20 = 0$

(3) $x^3 + 9x^2 + 63x + 98 = 0$

(4) $x^3 + 3x^2 + 21x + 38 = 0$

(5) $x^3 + 15x^2 + 60x + 176 = 0$

(6) $x^3 + 12x^2 + 36x + 8 = 0$

(7) $x^3 + 6x^2 + 6x - 8 = 0$

(8) $x^3 + 15x^2 + 63x + 73 = 0$

(9) $x^3 - 9x^2 + 8 = 0$

(10) $4x^3 - 9x^2 + 18x - 4 = 0$

(11) $2x^3 - 3x^2 + 6x - 9 = 0$

Solve the following equations by Ferrari's method.

(12) $x^4 - 4x^3 + 2x^2 + 12x - 15 = 0$

(13) $x^4 + 6x^3 + 4x^2 - 12x - 12 = 0$

(14) $x^4 - 6x^3 + 6x^2 + 6x - 7 = 0$

(15) $x^4 - 16x^3 + 59x^2 - 2x - 21 = 0$

(16) $x^4 - 3x^2 - 24x - 7 = 0$

(17) $x^4 - 18x^2 - 16x - 3 = 0$

(18) $x^4 - 17x^2 + 30x - 9 = 0$

(19) $4x^4 - 32x^3 + 47x^2 - 2x - 5 = 0$

(20) $9x^4 + 90x^3 + 198x^2 - 60x - 16 = 0$

(21) $16x^4 + 32x^3 + 23x^2 + 2x + 7 = 0$

Solve the following equations by Descartes' method.

(22) $x^4 - 2x^2 - 12x - 8 = 0$
(23) $x^4 - x^2 + 2x - 1 = 0$
(24) $x^4 + 8x^2 + 3x + 72 = 0$
(25) $x^4 - 13x^2 - 2x + 20 = 0$
(26) $x^4 + x^2 - \sqrt{2}\,x + 2 = 0$
(27) $x^4 - 14x^2 + 3\sqrt{5}\,x + 18 = 0$
(28) $x^4 - 12x^3 + 29x^2 - 52x - 70 = 0$
(29) $x^4 - 8x^3 + 23x^2 - 6x - 153 = 0$
(30) $x^4 + 8x^3 + 21x^2 + 20x + 12\sqrt{5}\,x + 24\sqrt{5} - 35 = 0$
(31) If one root of the following equation is 3, then find other four roots using Descartes' method.

$$x^5 - 3x^4 - 11x^3 + 37x^2 + 9x - 63 = 0$$

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