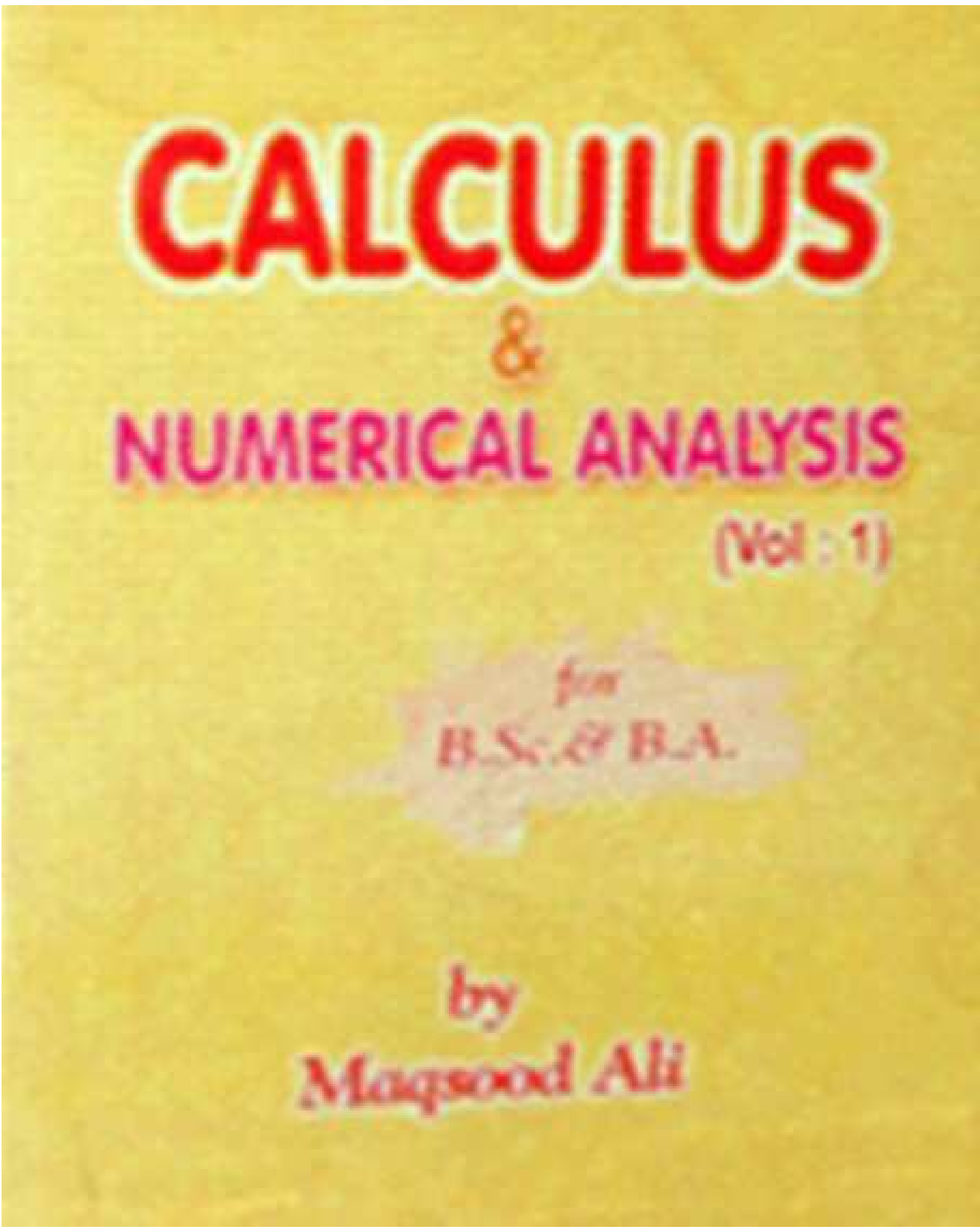
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
Maqsood Ali

Section D

NUMERICAL METHODS

ERRORS

ROUND OFF ERROR:

This error arises when the decimal fraction is rounded or chopped after the final digit.

Rounding off the numbers 1.2592367, 3.92876528, 9.33333333, 2.76359248 to three decimals, the numbers become 1.259, 3.929, 9.333 and 2.764. So that the numbers 1.259, 3.929, 9.333 and 2.764 are not true values, these are only approximate values. When we use these numbers in any calculation, the result have an error which is due to the rounding the numbers.

Rounding error is the most difficult to estimate.

TRUNCATION ERROR:

This error arises when an infinite process reduce into finite one.

For example some function have infinite number of terms in expansion, like that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To calculate the value of the function $f(x)$ for x we truncate it after some terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

These values have an error which is called truncation error.

Truncation error may be estimated and thus effectively reduced.

ABSOLUTE ERROR AND RELATIVE ERROR:

The absolute and relative errors are defined as

Absolute error =Calculated value − True value

$$\text{Relative error } = \frac{\text{Absolute error}}{\text{True value}}$$

The absolute error is used to find the error in calculation but relative error is a compares the two errors.

For example, suppose that two person A and Z have 100 and 100,000 rupees respectively. A counts them 110 rupees and Z 100010 rupees.

Absolute error for $A = 110 - 100 = 10$

Absolute error for $Z = 100010 - 100000 = 10$

$$\text{Relative error } A = \frac{\text{absolute error}}{\text{True value}} = \frac{10}{100} = 0.1$$

$$\text{Relative error } Z = \frac{\text{absolute error}}{\text{True value}} = \frac{10}{100000} = 0.0001$$

Above results shows that the absolute error are same for both A and Z, but relative error for Z is much less than relative error for A.

Therefore, relative error shows that A made a major mistake than Z.

ESTIMATION OF ERROR IN
(i) SUM (ii) PRODUCT (iii) QUOTIENT (iv) FUNCTION OF ONE VARIABLE:

(i) $y = x_1 \pm x_2 \pm x_3 \pm \cdots \pm x_n$

$If \pm \delta x_{1,} \pm \delta x_{2,} \pm \delta x_{3,} \cdots , \pm\delta_n$ be the errors in $x_1, x_2, x_3, \cdots , x_n$ respectively ,then corresponding error in y be $\pm\, \delta y$.

$$y \pm \delta y = (x_1 \pm \delta x_1) + (x_2 \pm \delta x_2) + (x_3 \pm \delta x_3) + \cdots + (x_n \pm \delta x_n)$$
$$\pm\delta y = \pm(\delta x_{1,} + \delta x_{2,} + \delta x_{3,} + \cdots + \delta x_n)$$

Absolute error in y

$$|\delta y| = |\delta x_1 + \delta x_2 + \delta x_3 + \cdots + \ \delta x_n|$$

Max. absolute error in y

$$|\delta y| \le |\delta x_1| + |\delta x_2| + |\delta x_3| + \cdots + |\delta x_n| = \sum_{i=1}^{n} |\delta x_n|$$

Hence absolute error in y is less than or equal to sum of absolute error in $x_1, x_2, x_3, \cdots , x_n$.

Maximum relative error:

$$\frac{|\delta y|}{|y|} \leq \sum_{i=1}^{n} \frac{|\delta x_n|}{|y|}$$

(ii) $y = x_1 x_2$

If $\pm\delta x_1$ and $\pm\delta x_2$ are the errors in x_1 and x_2 respectively, then the corresponding error in y is $\pm\delta y$.

$$y \pm \delta y = (x_1 \pm \delta x_1)(x_2 \pm \delta x_2)$$
$$\pm\delta y = \pm(\delta x_2)x_1 \pm (\delta x_1)x_2 \pm \delta x_1 \delta x_2$$

=Relative error in x_1 +Relative error in x_2

(iii) $y = \frac{x_1}{x_2} = x_1 x_2^{-1}$

$$y \pm \delta y = (x_1 \pm \delta x_1)(x_2 \pm \delta x_2)^{-1}$$
$$= \frac{1}{x_2}(x_1 \pm \delta x_1)\left[1 \pm \left(\frac{\delta x_2}{x_2}\right) \pm \cdots\right]^{-1}$$

Neglecting square and higher power of $\left(\frac{\delta x_2}{x_2}\right)$,we get

$$y \pm \delta y = \frac{1}{x_2}(x_1 \pm \delta x_1)\left(1 \pm \frac{\delta x_2}{x_2}\right)$$
$$y \pm \delta y = \frac{1}{x_2}\left(x_1 \pm \frac{x_1}{x_2}\delta x_2 \pm \delta x_1 \pm \frac{\delta x_1 \delta x_2}{x_2}\right)$$

Neglecting $\frac{\delta x_1 \delta x_2}{x_2}$

$$y \pm \delta y = \frac{x_1}{x_2} \pm \frac{x_1}{x_2^2}\delta x_2 \pm \frac{\delta x_1}{x_2}$$

$$\pm\delta y = \pm\left[\frac{x_1}{x_2^2}\delta x_2 + \frac{\delta x_1}{x_2}\right]$$

$$|\delta y| \leq \left|\frac{x_1}{x_2^2}\right||\delta x_2| + \left|\frac{1}{x_2}\right||\delta x_1|$$

Relative error in y:

$$\frac{|\delta y|}{|y|} \leq \frac{|\delta x_1|}{|x_1|} + \frac{|\delta x_2|}{|x_2|}$$

The relative error in product and quotient both are same.

So, if $y = x_1^{\pm}, x_2^{\pm}, \cdots x_n^{\pm}$ then the relative error in y is

$$\frac{|\delta y|}{|y|} \leq \frac{|\delta x_1|}{|x_1|} + \frac{|\delta x_2|}{|x_2|} + \cdots + \frac{|\delta x_n|}{|x_n|}$$

(iv) $y = f(x)$

Let $\pm\delta x$ be the error in x and the corresponding error in y be $\pm\delta y$.

$$y \pm \delta y = f(x \pm \delta x)$$

Applying Taylor's series

$$y \pm \delta y = f(x) \pm (\delta x) f'(x) \pm \frac{(\delta x)^2}{2!} f''(x) \pm \cdots$$

Neglecting square and higher power of (δx)

$$y \pm \delta x = f(x) \pm (\delta x) f'(x)$$

$$\pm\delta y = \pm(\delta x) f'(x)$$

$$|\delta y| = |\delta x|\,|f'(x)|$$

Relative Error:

$$\frac{|\delta y|}{|y|} = \frac{|\delta y||f'(x)|}{|f(x)|}$$

(v) $y = f(x_1, x_2, \cdots, x_n)$

Absolute error

$$|\delta y| = \left|\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_1} + \cdots + \delta x_n \frac{\partial f}{\partial x_n}\right|$$

Example 1:

The following quantities are in error as indicated

$$x = 50 \pm 0.05, y = 100 \pm 0.01, z = 300 \pm 0.4$$

Determine the maximum values of the absolute and relative errors in the result of each of the following operations.

(i) $x + 5y + z$ (ii) $In\,(x + y)$

Solution:-

(i) Let $f = x + 5y + z$

Maximum absolute error

$$|\delta f| = |\delta x| + 5|\delta y| + |\delta z| = 0.05 + 5(0.01) + 0.4 = 0.5$$

$$\text{Relative error} = \frac{|\delta f|}{|f|} = \frac{0.5}{850} = 5.88 \times 10^{-4}$$

(ii) Let $f = In(x + y)$

Maximum absolute error

$$|\delta f| = |\delta x|\left|\frac{\partial f}{\partial x}\right| + |\delta y|\left|\frac{\partial f}{\partial y}\right|$$

$$= (0.05)\left|\frac{1}{x+y}\right| + (0.01)\left|\frac{1}{x+y}\right| = 0.0004$$

Maximum relative error

$$\frac{|\delta f|}{|f|} = 0.0004$$

EXERCISE D-1

Q. 1 If $f = x_1^{\pm m_1}, x_2^{\pm m_2}, x_3^{\pm m_3}, \cdots, x_n^{\pm m_n}$

then show that the relative error in f is given by

$$\frac{|\delta f|}{|f|} \le |m_1|\left|\frac{\delta x_1}{x_1}\right| + |m_2|\left|\frac{\delta x_2}{x_2}\right| + \cdots + |m_n|\left|\frac{\delta x_n}{x_n}\right|$$

Q.2 The following quantities are in error as indicated.

$$x = 15 \pm 0.5, y = 50 \pm 0.6, z = 45 \pm 0.4$$

Determine the maximum value of the relative error in the result of each of the following.

(i) $x + 5y + 3z$ (ii) $5x - 3y + 2z$ (iii) xyz

(iv) xy/z (v) $ln(y - x)$ (vi) $\sqrt{xy}$

Q.3 Find the error in $sin 10^{\circ}30' \div \log_{10} 507.5$, assuming that the angle may be in error by 1' and that the number 507.5 may be in error 0.1.

Q.4 Find the error in the evaluation of the multiplication ($cos\ 70°15'$) ($sin\ 3°\ 20'$) assuming that the angles may be in error by 2' and 1' respectively .

AUTHOR

M. MAQSOOD ALI

ASSISTANT PROFESSOR OF MATHEMATICS

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BY

M. MAQSOOD ALI

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