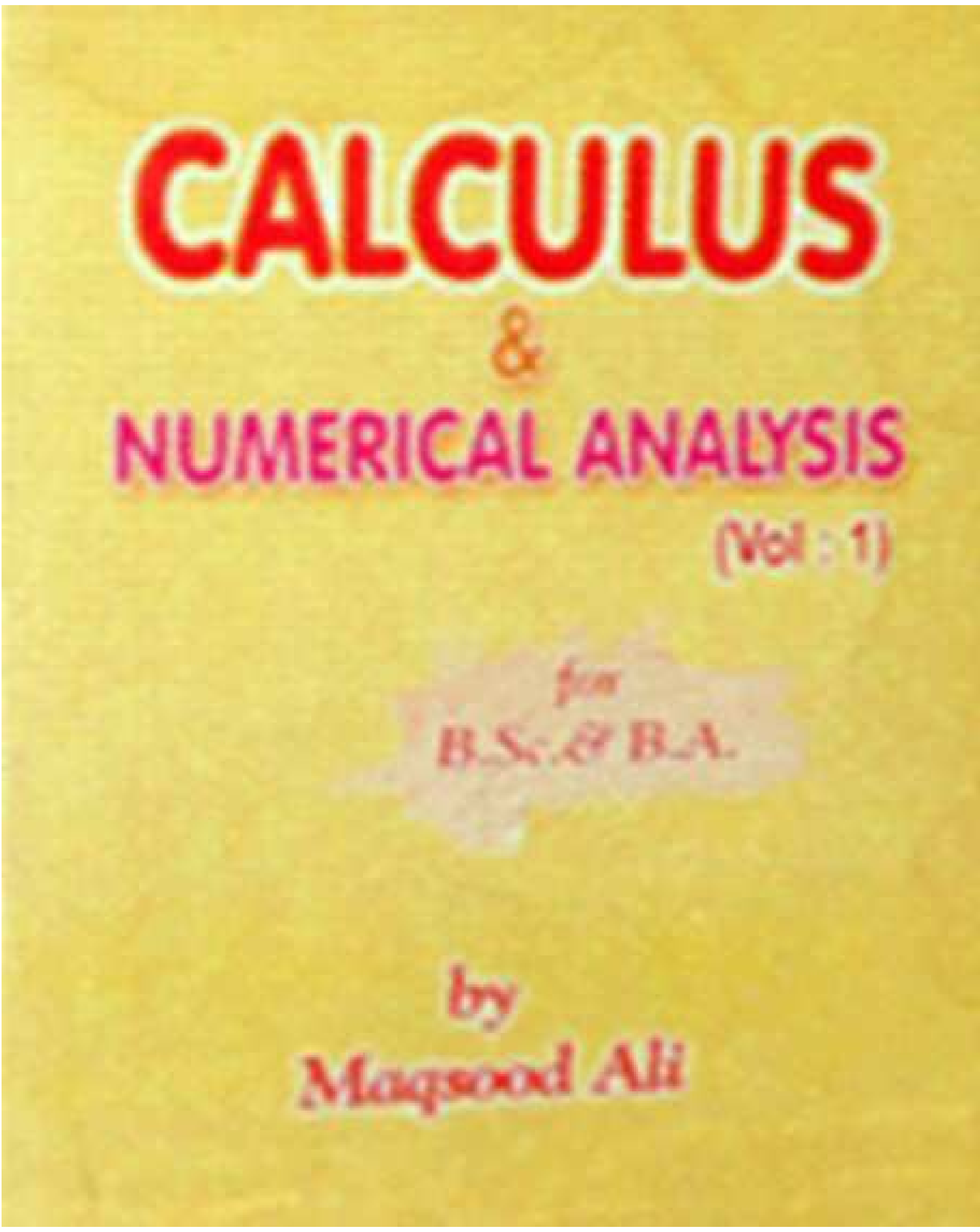
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
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TAYLOR`S SERIES

(1) If $f(x,y) \in C^n$ then

$$f(x,y) = \sum_{i=0}^{n-1} \frac{1}{i!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^i f(a,b) + R_n$$

$$f(x,y) = \sum_{i=0}^{n-1} \frac{1}{i!} D^i f(a,b) + R_n$$

where $D^i = \left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^i$

and $R_n = \frac{1}{n!} D^n f(a+\theta(x-a), b+\theta(y-b))$

(2) If $|x-a| \le |h|$ and $|y-b| \le |k|, f(x,y) \in C^n$ then

$$f(a+h, b+k) = \sum_{i=0}^{n-1} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(a,b) + R_n$$

when $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+\theta h, b+\theta k)$,
$0 < \theta < 1$

MACLAURIN'S SERIES

If $a = b = 0$

$$f(x,y) = \sum_{i=0}^{n-1} \frac{1}{i!}\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\}^i f(0,0) + R_n$$

$$= \sum_{i=0}^{n-1} \frac{1}{i!} D^i f(0,0) + R_n$$

where $R_n = \frac{1}{i!} D^n f(\theta x, \theta y)$, $0 < \theta < 1$

Proof Taylor's Series:

Let f be a real valued function and let $f \in C^{n+1}$

$$f: D \to R \quad , \quad D \subset R^2$$

Suppose that

$x = a + ht$ and $y = b + kt$

$$z = f(x, y) = f(a + ht\,, b + kt) = \phi(t)$$

$$\frac{dz}{dt} = \varphi^{(1)}(t) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) z$$

$$= Dz$$

$$\Rightarrow \frac{d}{dt} = D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$$

$$\frac{d^2z}{dt^2} = \varphi^2(t) = \frac{d}{dt}\left(\frac{dz}{dt}\right)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right)$$

$$= h^2\frac{\partial^2 z}{\partial x^2} + hk\frac{\partial^2 z}{\partial x\partial y} + hk\frac{\partial^2 z}{\partial y\partial x} + k^2\frac{\partial^2 z}{\partial y^2}$$

$$= h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial x\partial y} + k^2\frac{\partial^2 z}{\partial y^2}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 z$$

$$= D^2 z$$

Since $f \in C^{n+1} \Rightarrow \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$

Similarly

$$\frac{d^3z}{dt^3} = \varphi^3(t) = D^3 z$$

$$\frac{d^4z}{dt^4} = \varphi^4(t) = D^4 z \quad \text{and so on.}$$

Since $\varphi(t) = f(a + ht, b + kt)$

Hence

$$\varphi(0) = f(a, b)$$

$$\varphi^{(1)}(0) = Df(a, b)$$

$$\varphi^{(2)}(0) = D^2 f(a, b) \quad , \text{ and so on.}$$

φ is a function of of one variable t and $\varphi \in C^{n+1}$.
According to the Maclaurin's series for one variable,
$0 < \theta < 1$.

$$\varphi(t) = \varphi(0) + \frac{t}{1!}\varphi^{(1)}(0) + \frac{t^2}{2!}\varphi^{(2)}(0) + \cdots + R_n$$

$$f(a+ht, b+kt) = f(a,b) + \frac{t}{1!}Df(a,b) + \frac{t^2}{2!}D^{(2)}f(a,b) + \cdots + \frac{t^n}{n!}D^{(n)}f(a+\theta ht, b+\theta kt)$$

For $t = 1$

$$f(a+h, b+k) = f(a,b) + Df(a,b) + \frac{1}{2!}D^{(2)}f(a,b) + \cdots + \frac{1}{n!}D^n f(a+\theta h, b+\theta k)$$

$$= f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots$$

$$+ \frac{1}{n!} + \cdots + \frac{1}{n!}D^n f(a+\theta h, b+\theta k)^n f(a+\theta h, b+\theta k)$$

But $h = x - a$ and $k = y - b$

$$f(x,y) = f(a,b) + \left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}f(a,b) + \frac{1}{2!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^2 f(a,b) + \cdots + \frac{1}{n!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^n f\big(a+\theta(x-a), b+\theta(y-b)\big)$$

$$D = (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}$$

$$f(x,y) = f(a,b) + Df(a,b) + \frac{1}{2!}D^2 f(a,b) + \cdots + R_n$$

where $R_n = \frac{1}{n!}D^n f\big(a+\theta(x-a), b+\theta(y-b)\big)$

Proof Maclaurin's Series:

For $a = b = 0$

$$f(x,y) = f(0,0) + Df(0,0) + \frac{1}{2!}D^2 f(0,0) + \cdots + R_n$$

where $R_n = \frac{1}{n!}D^n f(\theta x, \theta y)$ and $D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

Example-1:

Expand $\cos(x+y)$

(i) in powers of x and y

(ii) in powers of $\left(x - \frac{\pi}{2}\right)$ and $\left(y - \frac{\pi}{4}\right)$

Solution:

$$f(x,y) = \cos(x+y)$$

n	$\frac{\partial^n}{\partial x^n}f(x,y)$	$\frac{\partial^n}{\partial y^n}f(x,y)$	$\frac{\partial^{n+1}}{\partial x^n \partial y}f(x,y)$	$\frac{\partial^{n+1}}{\partial x \partial y^n}f(x,y)$
1	$-\sin(x+y)$	$-\sin(x+y)$	$-\cos(x+y)$	$-\cos(x+y)$
2	$-\cos(x+y)$	$-\cos(x+y)$	$\sin(x+y)$	$\sin(x+y)$
3	$\sin(x+y)$	$\sin(x+y)$	$\cos(x+y)$	$\cos(x+y)$

n	$\frac{\partial^n}{\partial x^n}f(0,0)$	$\frac{\partial^n}{\partial y^n}f(0,0)$	$\frac{\partial^{n+1}}{\partial x^n \partial y}f(0,0)$	$\frac{\partial^{n+1}}{\partial x \partial y^n}f(0,0)$
1	0	0	-1	-1
2	-1	-1	0	0
3	0	0	1	1

$$f = f\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

n	$\frac{\partial^n}{\partial x^n}f$	$\frac{\partial^n}{\partial y^n}f$	$\frac{\partial^{n+1}}{\partial x^n \partial y}f$	$\frac{\partial^{n+1}}{\partial x \partial y^n}f$
1	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
2	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
3	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

(i) $f = f(0,0)\ \ and\ \ f(0,0) = 1$

$$Df = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0)$$

$$= x\frac{\partial}{\partial x}f(0,0) + y\frac{\partial}{\partial y}f(0,0) = 0$$

$$D^2 f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0)$$

$$= x^2\frac{\partial^2}{\partial x^2}f(0,0) + 2xy\frac{\partial^2}{\partial x\partial y}f(0,0) + y^2\frac{\partial^2}{\partial y^2}f(0,0)$$

$$= -x^2 - 2xy - y^2$$

$$= -(x+y)^2$$

$$D^3 f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) = 0$$

Taylor's series

$$f(x,y) = f(0,0) + Df(0,0) + \frac{1}{2!}D^2 f(0,0) + \frac{1}{3!}D^3 f(0,0)$$

$$+\frac{1}{4!}D^4 f(0,0) + \cdots$$

$$\cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \cdots$$

(ii) $f = f\left(\frac{\pi}{2},\frac{\pi}{4}\right)$ and $f = f\left(\frac{\pi}{2},\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$Df = \left\{(x-\frac{\pi}{2})\frac{\partial}{\partial x} + (y-\frac{\pi}{4})\frac{\partial}{\partial y}\right\}f\left(\frac{\pi}{2},\frac{\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}}\left\{\left(x-\frac{\pi}{2}\right) + (y-\frac{\pi}{4})\right\}$$

$$D^2 f = \left\{(x-\frac{\pi}{2})\frac{\partial}{\partial x} + (y-\frac{\pi}{4})\frac{\partial}{\partial y}\right\}^2 f\left(\frac{\pi}{2},\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\left\{\left(x-\frac{\pi}{2}\right) + (y-\frac{\pi}{4})\right\}^2$$

$$D^3 f = \left\{(x-\frac{\pi}{2})\frac{\partial}{\partial x} + (y-\frac{\pi}{4})\frac{\partial}{\partial y}\right\}^3 f\left(\frac{\pi}{2},\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\left\{\left(x-\frac{\pi}{2}\right) + (y-\frac{\pi}{4})\right\}^3$$

Taylor's series

$$f(x,y)=f\left(\frac{\pi}{2},\frac{\pi}{4}\right)+Df\left(\frac{\pi}{2},\frac{\pi}{4}\right)+\frac{1}{2!}D^2f\left(\frac{\pi}{2},\frac{\pi}{4}\right)$$
$$+\frac{1}{3!}D^3f\left(\frac{\pi}{2},\frac{\pi}{4}\right)+\cdots$$

$$\cos(x+y)=-\frac{1}{\sqrt{2}}\left[1+\frac{1}{1!}\left\{\left(x-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{4}\right)\right\}\right.$$
$$-\frac{1}{2!}\left\{\left(x-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{4}\right)\right\}^2$$
$$\left.-\frac{1}{3!}\left\{\left(x-\frac{\pi}{2}\right)\frac{\partial}{\partial x}+\left(y-\frac{\pi}{4}\right)\frac{\partial}{\partial y}\right\}^3+\cdots\right]$$

Example-2:

Expand e^{xy} in power of x and y.

Solution:

$$f(x,y)=e^{xy}$$

n	$\frac{\partial^n}{\partial x^n}f(x,y)$	$\frac{\partial^n}{\partial y^n}f(x,y)$	$\frac{\partial^{n+1}}{\partial x^n\partial y}f(x,y)$
1	ye^{xy}	xe^{xy}	$(xy+1)e^{xy}$
2	y^2e^{xy}	x^2e^{xy}	$(xy^2+2y)e^{xy}$
3	y^3e^{xy}	x^3e^{xy}	$(xy^3+3y^2)e^{xy}$

$\frac{\partial^{n+1}}{\partial x\partial y^n}f(x,y)$	$\frac{\partial^{n+2}}{\partial x^n\partial y^2}f(x,y)$	$\frac{\partial^{n+2}}{\partial x^2\partial y^n}f(x,y)$
$(xy+1)e^{xy}$	$(2x+x^2y)\,e^{xy}$	$(2y+xy^2)\,e^{xy}$
$(x^2y+2x)e^{xy}$	$(4xy+x^2y^2+2)e^{xy}$	$(4xy+x^2y^2+2)\,e^{xy}$
$(x^3y+3x^2)e^{xy}$	$(6xy+x^2y^3+6y)e^{xy}$	$(6x^2y+x^3y^2+6x)\,e^{xy}$

n	$\frac{\partial^n}{\partial x^n}f(0,0)$	$\frac{\partial^n}{\partial y^n}f(0,0)$	$\frac{\partial^{n+1}}{\partial x^n\partial y}f(0,0)$
1	0	0	$1=1!$
2	0	0	0
3	0	0	0

$\frac{\partial^{n+1}}{\partial x\partial y^n}f(0,0)$	$\frac{\partial^{n+2}}{\partial x^n\partial y^2}f(0,0)$	$\frac{\partial^{n+2}}{\partial x^2\partial y^n}f(0,0)$
$1=1!$	0	0
0	$2=2!$	$2=2!$
0	0	0

Since

$$\frac{\partial^n}{\partial^m x \partial^{n-m} y} f(0,0) = \begin{cases} 0 & when \quad n \neq 2m \\ m! & when \quad n = 2m \end{cases}$$

Using Taylor's series

$$e^{xy} = \sum_{i=0}^{n-1} \frac{1}{i!} (xy)^i + R_n$$

Exercise C-8

(1) Expand ,$\sin(x-y)$ in powers of

(i) x and y

(ii) $\left(x-\frac{\pi}{4}\right)$ and $(y-\frac{\pi}{2})$

(2) Expand $e^{ax} cosby$ in powers of

(i) x and y

(ii) $\left(x-\frac{1}{a}\right)$ and $(y-\frac{1}{b})$

(3) Expand $sinx\ cosy$ in powers of

(i) x and y

(ii) x and $(y-\frac{\pi}{4})$

(iii) $\left(x-\frac{\pi}{3}\right)$ and $(y-\frac{\pi}{6})$

(iv) $\left(x-\frac{\pi}{4}\right)$ and y

MAXIMA AND MINIMA

CRITICAL POINT:

(a) A point (x_0, y_0) in the domain of a $f(x, y)$ is a critical point of f if

(i) $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

(ii) f_x or f_y does not exist at (x_0, y_0).

(b) If (x_0, y_0) is a critical value for f, it is not the case that $f(x_0, y_0)$ must be either a relative maximum or a relative minimum. While the point (x_0, y_0) in the domain of f, where the function has a relative maximum or relative minimum, must be a **critical point.**

RELATIVE MAXIMUM, RELATIVE MINIMUM AND SADDLE POINTS:

A function of two variable $z = f(x, y)$ has a **relative maximum** at the point (x_0, y_0, z_0) if the surface $z = f(x, y)$ has a peak at (x_0, y_0, z_0) i.e $f(x, y) < f(x_0, y_0)$ for all values of x and y in the neighbourhood of (x_0, y_0).

Similarly, the function has a **relative minimum** at (x_0, y_0, z_0) if the surface $z = f(x, y)$ has a pit at (x_0, y_0, z_0) i.e $f(x, y) > f(x_0, y_0)$ for all values of x and y in the neighbourhood of (x_0, y_0).

Function $z = f(x, y)$ has a **saddle point** (x_0, y_0, z_0) if f is a relative minimum in one direction and a relative maximum in another direction, and hence f is neither relative maximum nor relative minimum at (x_0, y_0, z_0). At saddle point, the partial derivation f_x and f_y are both zero, but do not change sign.

DERIVATIVE TEST:

To find relative maxima, relative minima and saddle point for a function $z = f(x, y)$, proceed as follows.

(i) Compute f_x, f_y, f_{xx}, f_{yy} and f_{xy}

(ii) Solve simultaneously $f_x=0$ and $f_y=0$, to obtain all pair of numbers (critical points) (x_0, y_0) such that $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

(iv) Evaluate $\Delta(x, y) = f_{xx}\, f_{yy} - f_{xy}^2$. Then

(a) If $\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, the function has a relative maximum at (x_0, y_0, z_0).

(b) If $\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ or $f_{yy}(x_0, y_0) > 0$,

the function has a relative minimum at (x_0, y_0, z_0).
(c) If $\Delta(x_0, y_0) < 0$, the function has a saddle point at (x_0, y_0, z_0).
(d) Test fails if $\Delta(x_0, y_0) = 0$

ABSOLUTE MAXIMUM OR MINIMUM:

(i) A function $f(x, y)$ has an **absolute maximum** at a point (x_0, y_0) in the domain of f if

$$f(x_0, y_0) \geq f(x, y)$$

for all (x, y) in the domain of f
(ii) A function $f(x, y)$ has an **absolute minimum** at a point (x_0, y_0) in the domain of f if

$$f(x_0, y_0) \leq f(x, y)$$

for all (x, y) in the domain of f.

Example:

Find all critical points for the function.

$$f(x, y) = x^3 + y^3 - 3xy$$

Use derivative test to determine whether each is a maximum, minimum or a saddle point.

Solution:

$$f(x, y) = x^3 + y^3 - 3xy$$

The first and second partial derivatives are

$f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$
$f_{xx} = 6x$, $f_{yy} = 6y$
$f_{xy} = -3$, $f_{yx} = -3$

For critical points

$$f_x = 0 \text{ and } f_y = 0$$

$$3x^2 - 3y = 0 \quad \rightarrow (1)$$
$$3y^2 - 3x = 0 \quad \rightarrow (2)$$

Solving equations (1) and (2) simultaneously, we get the following critical points.

$(0, 0)$ and $(1, 1)$

$$\Delta(x, y) = f_{xx} f_{yy} - f_{xy}^2$$

$$= 9(4xy - 1)$$

$\Delta(0, 0) = -9 < 0$

∴The function has a saddle point (0, 0).

$\Delta(1, 1) = 27 > 0$, $f_{xx}(1, 1) = 6 > 0$, $f_{yy}(1, 1) = 6 > 0$

∴ The function has a relative minimum at (1, 1).

EXERCISE C- 9

Find relative maxima, relative minima and saddle point for the following function.

(1) $f(x,y) = xy\,(x + y - 1) - x^2$
(2) $f(x,y) = 4xy - 2x^2y - xy^2$
(3) $f(x,y) = \cos xy + \sin xy$
(4) The surface area of a rectangular box is 96 square units maximize the volume of the box.
(5) A rectangular box without a top is to have surface area 12 square meters. What are the dimensions of the box if it is to have maximum volume.
(6) The volume of a cylinder is 256 cubic feet, find the minimum area and dimensions of the cylinders.

EULER'S THEOREM

HOMOGENEOUS FUNCTION:

(a) A function $f(x,y)$ is called homogeneous function if degree n if it can be written in the form $x^{\mathrm{n}} f(y/x)$.
(b) A function *f(x, y)* is said to be homogeneous of degree n if

$$f(\lambda x, \lambda y) = \lambda^{\mathrm{n}} f(x,y)$$

where λ is a positive real number.

Example 1:

Show that $f(x,y) = x^2 + y^2 - xy\,\sin\,(y/x)$ is homogeneous of degree 2.

Solution:

(i) According to first def.
$f(x,y) = x^2[1 + (y/x)^2 + (y/x)\,\sin(y/x)] = x^2 f(y/x)$.
Hence $f(x,y)$ is homogeneous of degree 2.
(ii) According to second def.

$$f(\lambda x, \lambda y) = \lambda^2x^2 + \lambda^2y^2 + (\lambda x)\,(\lambda y)\,\sin\left(\frac{\lambda y}{\lambda x}\right)$$

$f(\lambda x, \lambda y) = \lambda^2\,[x^2 + y^2 + xy\,\sin\,(y/x)] = \lambda^2 f(x,y)$
$\therefore\ f(x,y)$ is homogeneous of degree 2.

EULER'S THEOREM:

If $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\,f(x,y)$$

Proof:

$f(x, y)$ is a homogeneous function of degree n, so

$$f(x,y) = x^n\,f(y/x)$$

$$\frac{\partial f}{\partial x} = n\,x^{n-1}\,f(y/x) + x^n.\frac{d}{d(\frac{y}{x})}\,f(y/x)\,\frac{\partial}{\partial x}\,(y/x)$$

$$\frac{\partial f}{\partial x} = n\,x^{n-1}\,f(y/x) - x^{n-2}yf'\,(y/x)$$

$$x\,\frac{\partial f}{\partial x} = nx^n\,f(y/x) - x^{n-1}\,y\,f'\,(y/x) \quad \rightarrow \text{(i)}$$

$$y\,\frac{\partial f}{\partial y} = x^{n-1}\,y\,f'\,(y/x) \quad \rightarrow \text{(ii)}$$

Adding equations (i) and (ii)

$$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} = nx^n\,f(y/x)$$

$$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} = n\,f(x,y)$$

Example 2:

If $u = tan^{-1}\frac{x^4 + y^4}{x^2 - y^2}$ show that

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = sin2u$$

Solution:

$$u = tan^{-1}\,\frac{x^4 + y^4}{x^2 - y^2}$$

$$tan\,u = \frac{x^4 + y^4}{x^2 - y^2}$$

$\frac{x^4 + y^4}{x^2 - y^2}$ is a homogeneous function of degree 2.

Hence *tan u* is also a homogeneous function of degree 2.

According to Euler's theorem.

$$x \frac{\partial}{\partial x} \tan u + y \frac{\partial}{\partial y} \tan u = 2 \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Example:

Verify Euler's theorem for $u = \cos^{-1}(y/x)$

Solution:

$u = \cos^{-1}(y/x)$ is a homogeneous function of degree zero.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \frac{\partial}{\partial x} \cos^{-1}(y/x) + y \frac{\partial}{\partial y} \cos^{-1}(y/x)$$

$$= x \frac{-1}{\sqrt{1-(y/x)^2}} \cdot \left(-\frac{y}{x^2}\right) + y \frac{-1}{\sqrt{1-(y/x)^2}} \left(\frac{1}{x}\right)$$

$= 0 \rightarrow (1)$

$$n\, f(x,y) = (0) . \cos^{-1}(y/x) = 0 \rightarrow (2)$$

By equation (1) and (2)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nf(x,y)$$

Euler's theorem is satisfied.

EXERCISE C - 10

Whether the following functions are homogeneous or not, if f is homogeneous then find its degree.

(1) $f(x,y) = \frac{x^3 + y^3}{x - y}$

(2) $f(x,y) = \sin \frac{x^2 + y^2}{x + y}$

(3) $f(x,y) = xy\sqrt{x^2 + y^2}$

(4) $f(x,y) = \dfrac{(x^2 + y^2)\, sinx}{x + y}$

(5) $f(x,y) = \dfrac{(x^3 - y^3)cos^{-1}\,(y/x)}{x^2 + y^2}$

Verify Euler's theorem for the following Ex: 6-9

(6) $f(x,y) = \cos^{-1}(y/x) + cot^{-1}\,(y/x)$

(7) $f(x,y) = (x^2 + y^2)\,(log\, y - log\, x)$

(8) $f(x,y) = e^{y/x} tan^{-1}\,(y\,/\,x)$

(9) $f(x,y) = (x^n + y^n)\, sin\,(y\,/\,x)$

(10) If $f(x,y) = u = sin^{-1}\dfrac{x^2 + y^2}{x + y}$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = tan\, u$$

(11) If $u = log\dfrac{x^4 + y^4}{x^2 + y^2}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$$

(12) If $u = sin^{-1}\dfrac{x^2\, y + y^3}{xy - y^2}$ prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = tan\, u$$

(13) If $u = log\,(x^2 + y^2) - log\,(x - y)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

(14) If $u = tan^{-1}\left(\dfrac{x^5 + y^5}{x + y}\right)^{1/2}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2\, u$$

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