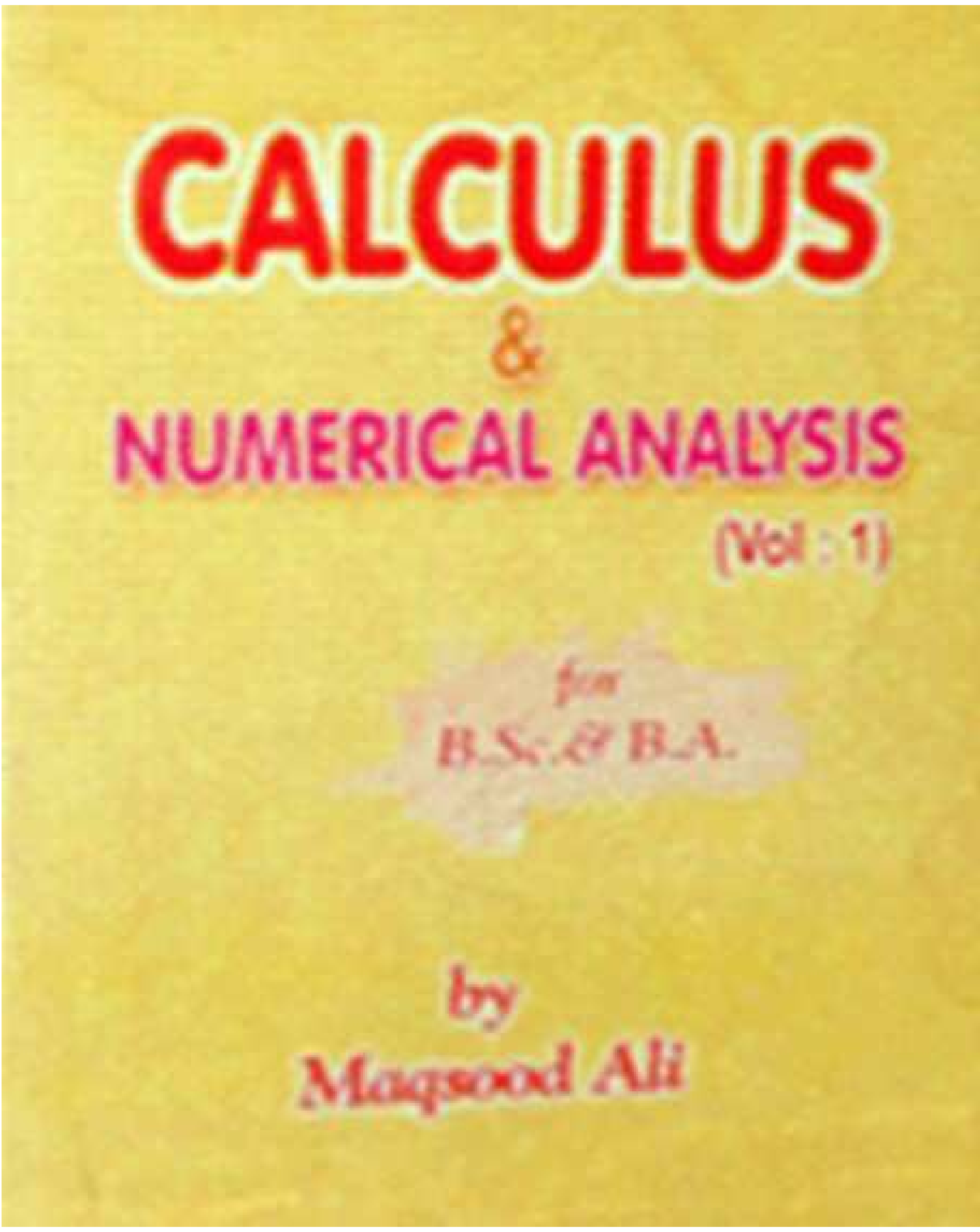
CALCULUS
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for
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by
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DIFFERENTIATION

A function $z = f(x,y)$ is said to be differentiable at (x,y) if there exist a relation of the form.

$$f = (x+\Delta x, y+\Delta y) - f(x,y) = A\Delta x + B\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

or

$$\Delta z = A\Delta x + B\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

where A and B are the function of x and y and ε_1 and ε_2 are the functions of Δx and Δy such that

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_1 = 0 \text{ and } \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_2 = 0$$

Difference Between Differentiation And Derivative:

Differentiability and derivability are two different operations. To understand the difference between these two operations consider a function of two variables $z = f(x,y)$.

"In forming partial derivative $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ changes Δx and Δy in x and y are consider separately".

For example, if we want to find the derivative of f with respect to x then y must be constant.

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

The derivative of f with respect to y holding x constant is given below.

$$\frac{\partial z}{\partial y} = \lim_{\Delta y\to 0}\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

But differentiation of $f(x,y)$ shows the effect of changing Δx and Δy in x and y together.

$$f(x+\Delta x, y+\Delta y) - f(x,y) = A\Delta x + B\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

Examples 1:

Using the definition of differentiation of a function $z = f(x,y)$.

that is

$$f(x+\Delta x, y+\Delta y) - f(x,y) = A\Delta x + B.\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

find A and B for the following function.

$$z = f(x,y) = x^2 + xy + y^2$$

and also show that $A = \frac{\partial z}{\partial x}$ and $B = \frac{\partial z}{\partial y}$

Solution:

$$f(x,y) = x^2 + xy + y^2$$

$$f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$= \{(x+\Delta x)^2 + (x+\Delta x)(y+\Delta y) + (y+\Delta y)^2\}$$
$$-(x^2 + xy + y^2)$$

$= x^2 + 2x\Delta x + (\Delta x)^2 + xy + x\Delta y + y\Delta x + \Delta x\, \Delta y + y^2$
$+2y\Delta y + (\Delta y)^2 - x^2 - xy - y^2$
$= 2x\Delta x + y\, \Delta x + x\Delta y + 2y\Delta y + (\Delta x)^2 + \Delta x. \Delta y + \Delta y^2$
$= (2x + y)\Delta x + (x + 2y)\Delta y + (\Delta x + \Delta y)\Delta x + \Delta y. \Delta y$
$\Rightarrow \varepsilon_1 = \Delta x + \Delta y$ and $\varepsilon_2 = \Delta y$

and $A = 2x + y = \dfrac{\partial z}{\partial x}$, $B = x + 2y = \dfrac{\partial z}{\partial y}$

THEOREM C -3:

A differentiable function is also derivable
or if $z = f(x, y)$ is differentiable then $A = \frac{\partial z}{\partial y}$ and $B = \frac{\partial z}{\partial y}$.

Proof:

Let $z = f(x, y)$ is a differentiable function.

$$\Delta z = A\Delta x + B\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \qquad \rightarrow (1)$$

we have to prove that f is derivable.

$z = f(x, y)$ is derivable mean partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ exist.

First suppose that there is no change in y that is $\Delta y = 0$
According to equation (1)

$$\Delta z = A\Delta x + \varepsilon_1\Delta x$$
$$A + \varepsilon_1 = \Delta z/\Delta x$$
$$A = \frac{\Delta z}{\Delta x} - \varepsilon_1$$

A is a function of x and y and ε_1 depends on Δx and Δy.
$\varepsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$

$$A = \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{\partial z}{\partial x}$$

Similarly,

$$B = \lim_{\Delta y \rightarrow 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{\partial z}{\partial y}$$

Hence f is derivable.

Example 2:

(3) Show that the following function is derivable but not differentiable at $(0,0)$.

$$f(x, y) = \frac{px^2 + qy^2}{x + y}, (x, y) \neq (0,0) \quad , f(0,0) = 0$$

Solution:

$$f_x(0,0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = p$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = q$$

Partial derivative of f exist, so f is derivable.

Differentiability:

$$f(\Delta x, \Delta y) - f(0,0) = A\Delta x + B\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\frac{p\Delta x^2 + q\Delta y^2}{\Delta x + \Delta y} = p\Delta x + q\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

taking $\Delta y = m\Delta x$

$$\frac{p + qm^2}{1+m} = p + qm + \varepsilon_1 + \varepsilon_2 \mathrm{m}$$

$$\mathrm{L.H.S} \to \frac{p+qm^2}{1+m}$$

and $$\mathrm{R.H.S} \to p + qm \neq \frac{p+qm^2}{1+m}$$

Hence $f(x,y)$ is not differentiable at $(0,0)$.

Example 3:

Whether the following function f is differentiable or not at $(0,0)$.

$$f(x,y) = \sqrt{|xy|}$$

Solution:

$$A = f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

$$B = f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta x} = 0$$

Partial derivatives of f exist.

Differentiability at $(0,0)$:

$$f(0+\Delta x, 0+\Delta y) - f(0,0) = \Delta x A + \Delta y . B + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\sqrt{|\Delta x . \Delta y|} - 0 = 0 + 0 + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\sqrt{|\Delta x . \Delta y|} = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Let $\Delta y = m . \Delta x$

$$\sqrt{|\Delta x . m\Delta x|} = \varepsilon_1 \Delta x + \varepsilon_2 (m\, \Delta x)$$

$$\sqrt{|m|} = \varepsilon_1 + m\varepsilon_2$$

$\mathrm{R.H.S} \to 0$

But $\mathrm{L.H.S} \to \sqrt{|m|} \neq 0$

Therefore $f(x,y)$ is not differentiable at $(0,0)$.

Example-4:

Discuss the differentiability of f at $(0,0)$

$f(x,y) = |x| + |y|$

Solution:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} = \pm 1$$

{because $\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$

and $\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$}

Similarly $f_y(0,0) = \pm 1$.

Partial derivatives of f do not exist at $(0,0)$.

Hence $f(x,y)$ is not differentiable at $(0,0)$.

Theorem C-4:

If $f(x,y)$ is differentiable at (a,b) it is continuous there.

Taking an example show that converse is not true.

Proof:

Since $f(x,y)$ is differentiable at (a,b), so

$$f(x + \Delta x, y + \Delta y) - f(a,b) = A\Delta x + B\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $\Delta x \to 0, \Delta y \to 0$

Since $f(x,y)$ is differentiable, so the partial derivatives of f exist and do not depend on Δx or Δy, therefore

$A\,\Delta x = f_x(a,b).\Delta x \to 0$ as $\Delta x \to 0$

and $B\,\Delta y = f_y(a,b).\Delta y \to 0$ as $\Delta y \to 0$

Then we can write

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(x + \Delta x, y + \Delta y) - f(a,b) = 0$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(x + \Delta x, y + \Delta y) = f(a,b)$$

Hence $f(x,y)$ is continuous at (a,b).

Converse:

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}$$

is continuous at $(0,0)$ but not differentiable there.

Exercise C-5

Show that the following functions are derivable but not differentiable at $(0,0)$.

(1) $$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{for} \quad (x,y)\neq(0,0)\\ 0 & \text{for} \quad (x,y)=(0,0)\end{cases}$$

(2) $$f(x,y)=\begin{cases}\dfrac{ax^2+by^2}{x+y} & \text{for} \quad (x,y)\neq(0,0)\\ 0 & \text{for} \quad (x,y)=(0,0)\end{cases}$$

(3) Show that the following function is differentiable at $(0,0)$

$$f(x,y)=\begin{cases}\dfrac{x^6-2y^6}{x^2+y^2} & \text{for} \quad (x,y)\neq(0,0)\\ 0 & \text{for} \quad (x,y)=(0,0)\end{cases}$$

DIFFERENTIAL

If $z = f(x, y)$ be a function of two variables then dz is called differential such that

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

In general, if $z = f(x_1, x_2, \cdots, x_n)$ then

$$dz = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \cdots + \frac{\partial z}{\partial x_n}dx_n$$

An application of differential:

An important application of the differential is the assessment of the impact that an error in measurement may have on the computed value of a function.

Example 1:

The length x of a rectangle is measured to be 100 inches with a possible error of 0.5 inch and the width y is measured to be 40 inches with a possible error of 0.15 inch. Estimate the largest error in the computed area of the rectangle.

Solution:

Since $x = 100,\ y = 40,\ dx = 0.5$ and $dy = 0.15$

The area function is

$$A(x, y) = xy$$

$$\frac{\partial A}{\partial x} = y = 40$$

$$\frac{\partial A}{\partial y} = x = 100$$

The differential or the error in the area is

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$

$$= 40\,(0.5) + (100)(0.15)$$

$= 35$ square inchs.

Example 2:

Using differential estimate $\sqrt[3]{67}\ \sqrt{10}\ \sqrt[5]{31}$

Solution:

$$\sqrt[3]{67}\sqrt{10}\ \sqrt[5]{31} = \sqrt[3]{64+3}\ \sqrt{9+1}\ \sqrt[5]{32-1}$$

$$= \sqrt[3]{x+dx}\ \sqrt{y+dy}\ \sqrt[5]{z+dz} \quad \rightarrow (i)$$

$x = 64, dx = 3; y = 9, dy = 1, z = 32, dz = -1$

Consider a function f

$$f(x,y,z) = \sqrt[3]{x}\ \sqrt{y}\ \sqrt[5]{z} = x^{1/3}\, y^{1/2}\, z^{1/5} \quad \rightarrow (ii)$$

$$\frac{\partial f}{\partial x} = \frac{y^{1/2}z^{1/5}}{3x^{2/3}} = \frac{9^{1/2}32^{1/5}}{3(64)^{2/3}} = \frac{1}{8}$$

$$\frac{\partial f}{\partial y} = \frac{x^{1/3}z^{1/5}}{2y^{1/2}} = \frac{(64)^{1/3}(32)^{1/5}}{2(9)^{1/2}} = \frac{4}{3}$$

$$\frac{\partial f}{\partial z} = \frac{x^{1/3}y^{1/2}}{5z^{4/5}} = \frac{(64)^{1/3}(9)^{1/2}}{5(32)^{4/5}} = \frac{3}{20}$$

The differential df is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \frac{1}{8}(3) + \frac{4}{3}(1) + \frac{3}{20}(-1)$$

$$= \frac{3}{8} + \frac{4}{3} - \frac{3}{20} = \frac{187}{120}$$

Since

$$f(x,y,z) = \sqrt[3]{x}\ \sqrt{y}\ \sqrt[5]{z}\ = \sqrt[3]{64}\ \sqrt{9}\ \sqrt[5]{32} = 24$$

Hence

$$\sqrt[3]{67}\ \sqrt{10}\ \sqrt[5]{31} = f + df = 24 + \frac{187}{120}$$

$$= \frac{3067}{120} = 25.558$$

Exercise C-6

Use differential to estimate the following:

(1) $\sqrt[3]{9}.\sqrt[5]{30}$

(2) $\sqrt[4]{80}.\sqrt[3]{62}.\sqrt{5}$

(3) $\ln(3.1^3 + 7.2^2 - 0.8^4)$

Use differential to estimate maximum error that can arise from calculating.

(4) $f(x, y) = xy^3 - 8x^2y^2$

if $x = 1 \pm 0.01$ and $y = 8 \pm 0.2$

(5) $f(x, y) = 3x^2y - 5xy$

if $x = 1 \pm 0.01$ and $y = 8 \pm 0.2$

(6) Radius of a circular cone is increased by 0.5% and the height is decreased by 0.1% find the percentage change in the volume.

(7) A quantity z is to be calculated from the formula $z = (x - y)/(x + y)$, assuming that $x = 25$ with a possible error 0.5 and $y = 50$ with a possible error 0.2, using differential calculate error in z.

(8) Suppose that the height of a right circular cylinder changes from 10 to 10.4 and radius changes 5 to 5.2 estimate the change in volume of the cylinder.

DIFFERENTIATION AND DIFFERENTIAL

Geometric significance of Δz and dz:

$z = f(x, y)$ is a function of the surface and $g(x, y)$ is the function of the tangent plane on the surface at the point $\left(a, b, f(a, b)\right)$,as shown in the figure. The values of f and g are same at (a, b) that is $f(a, b) = g(a, b)$.

(a, b) and $(a + \Delta x, b + \Delta y)$ are two points on the xy-plane and Δx and Δy are two small real numbers, so that $\left(a, b, f(a, b)\right)$ and $(a + \Delta x, b + \Delta y, f(a + \Delta x, b + \Delta y))$ are two points on the surface $f(x, y)$.

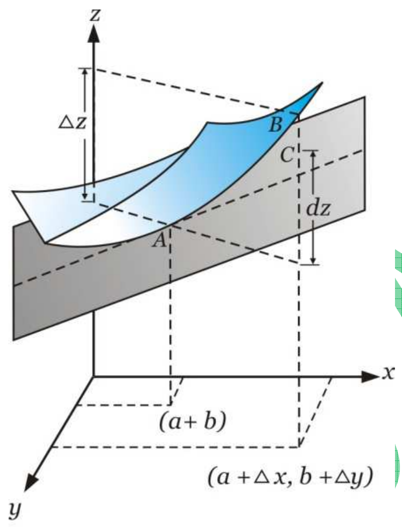

Figure C-18

Δz is the difference between two values $f(a + \Delta x, b + \Delta y)$ and $f(a, b)$ on the surface $f(x, y)$.

$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) \quad \rightarrow (1)$

It can be written as

$\Delta z = \Delta x. f_x(a, b) + \Delta y. f_y(a, b) + \Delta x. \varepsilon_1 + \Delta x. \varepsilon_2 \rightarrow (2)$

$\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$

According to equation (1) and equation (2)

$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$

$= \Delta x. f_x(a, b) + \Delta y. f_y(a, b) + \Delta x. \varepsilon_1 + \Delta x. \varepsilon_2$

Differential dz is the difference between two values $g(a + \Delta x, b + \Delta y)$ and $g(a, b)$ where $(a, b, g, (a, b))$ and $(a + \Delta x, b + \Delta y, g(a + \Delta x, b + \Delta y))$ are the two points on the tangent plane.

$dz = g(a + \Delta x, b + \Delta y) - g(a, b) \quad \rightarrow (3)$

It can be written as

$dz = \Delta x. f_x(a, b) + \Delta y. f_y(a, b) \quad \rightarrow (4)$

According to equations (3) and (4)

$dz = g(a + \Delta x, b + \Delta y) - g(a, b)$

$= \Delta x. f_x(a, b) + \Delta y. f_y(a, b)$

So Δz is approximately equal to differential dz $(i.e\ \Delta z \cong dz)$. The accuracy of this approximation increases as Δx and Δy become smaller.

Example:

The function f of a surface is defined by

$z = f(x,y) = 9 - \sqrt{9-(x-1)^2-(y-2)^2}$

If $x + \Delta x = 2 + 0.5$ and $y + \Delta y = 3 + 0.7$,then

(a) Find Δz

(b) Find differential dz

and also show that if $g(x,y)$ be the function of the tangent plane on the surface $f(x,y)$ then

$g(x+\Delta x, y+\Delta y) - g(x,y) = dz$

Solution:

$f(x,y) = 9 - \sqrt{9-(x-1)^2-(y-2)^2}$

$f(x,y) = 9 - \sqrt{-x^2-y^2+2x+4y+4} \quad \rightarrow (1)$

For $x = 2$ and $y = 3$

$f(2,3) = 9 - \sqrt{-(2)^2-(3)^2+2(2)+4(3)+4}$

$f(2,3) = 9 - \sqrt{7}$

By equation (1)

$f(x+\Delta x, y+\Delta y)$

$= 9 - \sqrt{-(x+\Delta x)^2-(y+\Delta y)^2+2(x+\Delta x)+4(y+\Delta y)+4}$

$f(2.5,3.7) = 9 - \sqrt{-(2.5)^2-(3.7)^2+2(2.5)+4(3.7)+4}$

$f(2.5,3.7) = 9 - \sqrt{3.86}$

(a) Δz:

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$= f(2.5,3.7) - f(2,3)$

$= \left(9-\sqrt{3.86} - (9-\sqrt{7})\right)$

$= \sqrt{7} - \sqrt{3.86} = 0.68$

(b) (i) $dz = \Delta x . f_x(x,y) + \Delta y . f_y(x,y)$

$f(x,y) = 9 - \sqrt{-x^2-y^2+2x+4y+4}$

Partial derivatives with respect to x and y.

$f_x(x,y) = \dfrac{x-1}{\sqrt{-x^2-y^2+2x+4y+4}}$

$f_y(x,y) = \dfrac{y-2}{\sqrt{-x^2-y^2+2x+4y+4}}$

when $x = 2$ and $y = 3$.

$f_x(2,3) = 1/\sqrt{7}$ and $f_y(2,3) = 1/\sqrt{7}$

Differential df:

$$df = \Delta x . f_x(2,3) + \Delta y . f_y(2,3)$$
$$= (0.5). 1/\sqrt{7} + (0.7)1/\sqrt{7}$$
$$= (6/5)\sqrt{7} = 0.45$$

(ii) Let g be the function of tangent plane on the surface$f(x, y)$ at the point (2,3)

$$g(x, y) = (x - x_o)f_x(x_o, y_o) + (y - y_o)f_y(x_o, y_o)$$
$$+f(x_o, y_o)$$
$$g(x, y) = (x - 2)f_x(2,3) + (y - 3)f_y(2,3) + f(2,3)$$
$$= (x - 2)(1/\sqrt{7}) + (y - 3)(1/\sqrt{7}) + 9 - \sqrt{7}$$
$$= 1/\sqrt{7}(x + y + 9\sqrt{7} - 12)$$

when $x = 2$ and $y = 3$

$$g(2,3) = 1/\sqrt{7}(2 + 3 + 9\sqrt{7} - 12)$$
$$= \frac{9\sqrt{7} - 7}{\sqrt{7}}$$
$$g(x + \Delta x, y + \Delta y) = 1/\sqrt{7}\{(x + \Delta x) + (y + \Delta y) + 9\sqrt{7} - 12\}$$
$$g(2.5,3.7) = 1/\sqrt{7}\{2.5 + 3.7 + 9\sqrt{7} - 12\}$$
$$= \frac{9\sqrt{7} - 5.8}{\sqrt{7}}$$

Differential dz:

$$g(x + \Delta x, y + \Delta y) - g(x, y)$$
$$= g(2.5,3.7) - g(2,3)$$
$$= \frac{9\sqrt{7} - 5.8}{\sqrt{7}} - \frac{9\sqrt{7} - 7}{\sqrt{7}}$$
$$= \frac{6}{5\sqrt{7}} = 0.45 = dz$$

Exercise C-7

For the following function f

(a) Find Δf

(b) Find differential df

and also show that if $g(x,y)$ be the function of the tangent plane on the surface $f(x,y)$ then

$$g(x+\Delta x, y+\Delta y) - g(x,y) = dz$$

(1) $f(x,y) = 3x^2 + y^2$

$x + \Delta x = 3 + 0.05$, $y + \Delta y = 5 + 0.1$

(2) $$f(x,y) = \frac{xy}{\sqrt{x^2+y^2+xy}}$$

$x + \Delta x = 2 - 0.01$, $y + \Delta y = 4 + 0.05$

(3) $f(x,y) = x^2 - y^2 + 3xy$

$x + \Delta x = 2 - 0.5, y + \Delta y = 3 - 0.01$

(4) $f(x,y) = x^2 + 2y^2$

$x + \Delta x = 8 + 0.2$, $y + \Delta y = 7 - 0.1$

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