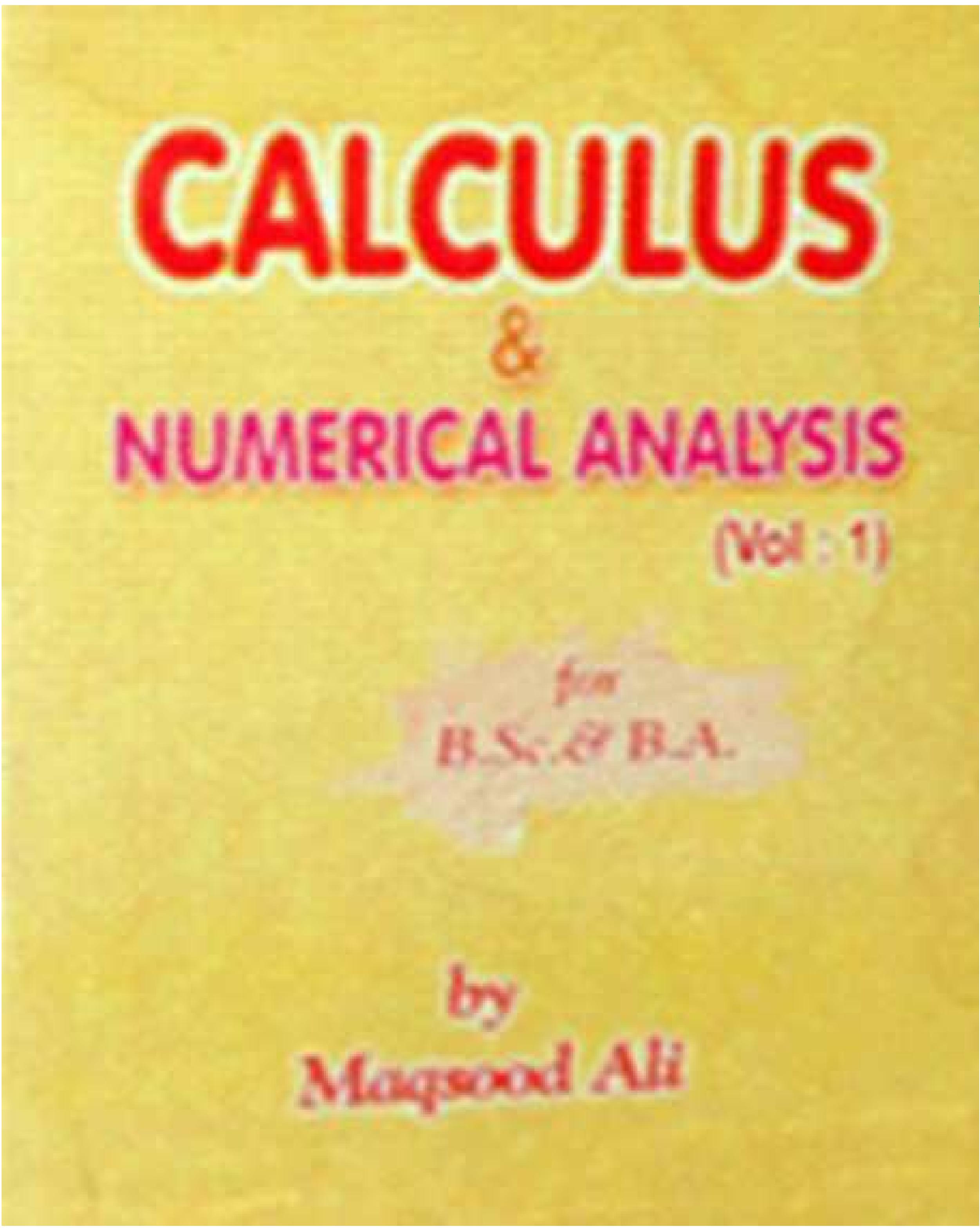
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
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PARTIAL DERIVATIVES

Partial derivatives of a function of several variables are ordinary derivatives with respect to a variable, holding all other variables constant.

If $f(x_1, x_2, \cdots, x_n)$ is a function of n variables then the partial derivative of f with respect to x_1 can be defined as

$$\frac{\partial f}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)}{\Delta x_1}$$

Similarly we can define $\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \cdots, \frac{\partial f}{\partial x_n}$

If $f(x, y, z)$ is a function of three variables. We can obtain a function of one variable by assigning fixed values to the other two variables. If y and z are regarded as fixed the derivative of f with respect to x is called partial derivative of f w.r.t x. This partial derivative is denoted by $\partial f/\partial x$ or $f_x(x, y)$ similar notations are used for the partial derivatives with respect to y and z.

If y and z are fixed values and x is a variable then the derivative of f can be defined as

$$f_x(x, y, z) = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

If x and z are fixed values and y is a variable then the derivative of f w.r.t. y is

$$f_y(x, y, z) = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

If x and y are fixed values and z is a variable then the derivative of f w.r.t .z is

$$f_z(x, y, z) = \frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

GEOMETRIC SIGNIFICANCE OF PARTIAL DERIVATIVES:

1- Three Dimensional Coordinate System:

Consider a three dimensional coordinate system as shown in figure.

2- Surface $f(x, y)$:

The graph of the function $z = f(x, y)$ is a surface in a three dimensional coordinates system.

3- Planes $y = b$ and $x = a$:

Consider a point (a, b) on the xy-plans.The plan $y = b$ is parallel to xz −plane and the plane $x = a$ is parallel to yz-plane as shown in the figures.

4- Curves $f(x, b)$ and $f(a, y)$ on the surface:

The intersection of the surface $f(x, y)$ and the plane $y = b$ forms a curve $f(x, b)$.

Similarly the intersection of the surface $f(x, y)$ and the plane $x = a$ forms a curve $f(a, y)$ as shown in the figure.

5- Partial Derivatives $f_x\,(a, b)$ and $f_y(a, b)$:

Partial derivative $f_x\,(a, b)$ is the slope of the tangent line to the curve $f(x, b)$ at the point $(a, b, f(a, b)$. So that $f_x\,(a, b)$ is the slope of the line parallel to xz-plane and tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$, as shown in the

figure C-17.

Example-1:

Find the derivative of the following function by first principle

$$f(x, y) = x^2y + 2y^2$$

Solution:

$$\frac{\partial}{\partial x} f(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 y + 2y^2 - (x^2y + 2y^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2y + 2xy\Delta x + (\Delta x)^2 y + 2y^2 - x^2y - 2y^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2xy + \Delta xy)\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2xy + \Delta xy) = 2xy$$

$$\frac{\partial}{\partial y} f(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[x^2(y + \Delta y) + 2(y + \Delta y)^2] - (x^2y + 2y^2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2y + x^2\Delta y + 2y^2 + 4y\Delta y + (\Delta y)^2 - x^2y - 2y^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 + 4y + \Delta y)\Delta y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (x^2 + 4y + \Delta y)$$

$$= x^2 + 4y$$

CHAIN RULE

(a) If $F = f(u, v)$ and u and v are functions of two variables $u = u(x, y)$ and $v = v(x, y)$ then

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial y}$$

(b) If $F = f(u, v, w)$ and u, v and w are functions of three variables $u = u(x, y, z)$ and $v = v(x, y, z)$ and $w = w(x, y, z)$, then

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}.\frac{\partial w}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial y} + \frac{\partial F}{\partial w}.\frac{\partial w}{\partial y}$$

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial z} + \frac{\partial F}{\partial w}.\frac{\partial w}{\partial z}$$

Example:

If $F = f(u, v) = 2u^3 + 3v$ and $u = u(x, y) = 5x^2 + 10y$ and $v = v(x, y) = x^2y$ then find $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.

Solution:

$$\frac{\partial F}{\partial u} = 6u^2 = 6(5x^2 + 10y)^2 \quad , \quad \frac{\partial F}{\partial v} = 3$$

$$\frac{\partial u}{\partial x} = 10x \quad , \quad \frac{\partial u}{\partial y} = 10$$

$$\frac{\partial v}{\partial x} = 2xy \quad , \quad \frac{\partial v}{\partial y} = x^2$$

The chain rule

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial x}$$
$$= 6(5x^2 + 10y)^2.10x + 3(2xy)$$
$$= 60x(5x^2 + 10y)^2 + 6xy$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}.\frac{\partial v}{\partial y}$$
$$= 6(5x^2 + 10y)^2.10 + 3(x^2)$$
$$= 60(5x^2 + 10y)^2 + 3x^2$$

Exercise C-3:

Find the partial derivatives of the following functions by first principle.

(1) $Z = f(x,y) = 2x^2 + xy$ (2) $f(x,y) = \frac{x}{y}$

(3) $f(x,y) = y\sin x$

(4) Find the partial derivatives for the following function at $(\mathbf{0},\mathbf{0},\mathbf{0})$

$$f(x,y,z) = \frac{xyz}{\sqrt{x^2+y^2+z^2}}$$

Such that $(x,y,z) \neq (0,0,0)$ and $f(0,0,0) = 0$

(5) If $F = f(x,y) = y\sin x + e^{xy} - x^2y^2$,
then find

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x \partial y}, \frac{\partial^2 F}{\partial y \partial x}. \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}$$

(6) If $F = f(x,y) = y^3 \tan x + tan^{-1}\left(\frac{y}{x}\right) + 3xy$,
then find

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x \partial y}, \frac{\partial^2 F}{\partial y \partial x}. \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}$$

(7) If $F = f(x,y) = 3x^2y + x^2yz\tan(xz)$, then find

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial^2 F}{\partial x \partial y}, \frac{\partial^2 F}{\partial x \partial z}, \frac{\partial^2 F}{\partial y \partial z}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial z^2}, \frac{\partial^3 F}{\partial z^3}$$

(8) If $f(x,y) = 6xz^4 + 5y^3z + e^{xyz}$,then find

$$\frac{\partial^3 F}{\partial x^3}, \frac{\partial^3 F}{\partial y^3}, \frac{\partial^3 F}{\partial z^3}$$

(9) If $F = f(u,v,w) = In(u^2 - 3v + w^3)$ where
$u = \sqrt{x} + 3y, v = ylnx + y^2,\ w = 2x,$
then find

$$\left[\frac{\partial F}{\partial x}\right]_{(1,3)} \text{ and } \left[\frac{\partial F}{\partial y}\right]_{(1,3)}$$

(10) If $F = f(u,v,w) = uv - vw + uw$ where
$u = 3x + y,\ v = xyz + 5,\ w = 6 + z^2$, then find

$$\left[\frac{\partial F}{\partial x}\right]_{(1,1,2)} \left[\frac{\partial F}{\partial y}\right]_{(1,1,2)} \text{ and } \left[\frac{\partial F}{\partial z}\right]_{(1,1,2)}$$

(11) If $F = f(u,v,w) = \dfrac{u+v}{w}$, where

$u = lnx + y + z$, $v = xz + y$, $w = y^2 + 6yz$,

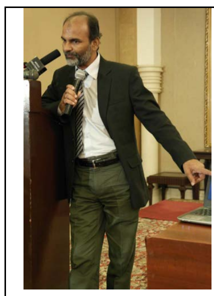
then find

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \text{ and } \frac{\partial F}{\partial z} \quad \text{at } (1,2,-1)$$

(12) If f is a function of polar coordinates r and θ where $x = r\cos\theta$ and $= r\sin\theta$,

then show that

$$\left[\frac{\partial F}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial F}{\partial \theta}\right]^2 = \left[\frac{\partial F}{\partial x}\right]^2 + \left[\frac{\partial F}{\partial y}\right]^2$$



TANGENT PALNE

FUNCTION OF THE TANGENT PLANE :

The graph of the function $f(x, y)$ is a surface in a three dimensional coordinate system.

$g(x, y)$ is the function of the tangent plane at the point (x_o, y_o, z_o) on the surface $f(x, y)$.

$g(x, y) = (x - x_o)\, f_x\, (x_o, y_o) + (y - y_o)\, f_y\, (x_o, y_o) + f(x_o, y_o)$

EQUATION OF THE TANGENT PLANE :

(i) $z = f(x, y)$ is an equation of the surface , the equation of the tangent plane at the point (x_o, y_o, z_o) is given below.

$(x - x_o)\, f_x\, (x_o, y_o) + (y - y_o)\, f_y\, (x_o, y_o) - (z - z_o) = 0$

(ii) $F(x, y, z) = 0$ is the implicit equation of the surface, the equation of the tangent plane at the point (x_o, y_o, z_o) on the surface is

$$(x - x_o) f_x(x_o, y_o, z_o) + (y - y_o) f_y(x_o, y_o, z_o) + (z - z_o) f_z(x_o, y_o, z_o) = 0$$

Example 1:

If f is the function of a surface such that $f(x, y) = x^2 + y^2$, then find the function of the tangent plane at (0,0,0).

Solution:

$$f(x, y) = x^2 + y^2 \quad \rightarrow (1)$$

$f_x(0,0) = 0$ and $f_y(0,0) = 0$

$g(x, y)$ is the function of the tangent plane at (0,0,0).

$$g(x, y) = (x - 0) f_x(0,0) + (y - 0) f_y(0,0) + f(0,0)$$

$$g(x, y) = 0$$

Example 2 :

$z = 12 - 3x^2$ is the equation of the surface. Find the equation of the tangent plane at (1,6,9).

Solution:

$$z = f(x, y) = 12 - 3x^2 \quad \rightarrow (1)$$

$f_x(1,6) = -6 \quad , \quad f_y(1,6) = 0$

The equation of the tangent plane.

$$(x - x_0) f_x(x_o, y_o) + (y - y_o) f_y(x_o, y_o) - (z - z_o) = 0$$

$$(x - 1)(-6) + 0 - (z - 9) = 0$$

$$6x + z - 15 = 0 \quad \rightarrow (2)$$

Equation (2) is the equation of the tangent plane at (1,6,9).

Example 3:

$x^2 + xz^2 - yz = 0$ is the equation of the surface , find the equation of the tangent plane at (1,2,1).

Solution:

$$f(x,y,z) = x^2 + xz^2 - yz = 0$$

$f_x(x,y,z) = 2x + z^2 \quad \Rightarrow \quad f_x(1,2,1) = 3$

$f_y(x,y,z) = -z \quad \Rightarrow \quad f_y(1,2,1) = -1$

$f_z(x,y,z) = 2xz - y \quad \Rightarrow \quad f_z(1,2,1) = 0$

The equation of the tangent plane

$$(x-1)f_x(1,2,1) + (y-2)f_y(1,2,1) + (z-1)f_z(1,2,1) = 0$$

$$(x-1)(3) + (y-2)(-1) + (z-1)(0) = 0$$

$$3x - y - 1 = 0$$

EXERCISE C-4

Find the function of the tangent plane for the following at the indicated point.

(1) $f(x,y) = 3xy + x^2$ $(3,1,19)$

(2) $f(x,y) = x^2 + y^2 + xy$ $(1,1,3)$

(3) $f(x,y) = x^3 + xy^2 + y^3$ $(1,1,3)$

Find the equation of the tangent plane for the following at the indicated point.

(4) $z = x^2 + y^2$ $(1,2,5)$

(5) $z = 5x^3 + 3y$ $(1,1,8)$

(6) $z = xy$ $(2,3,6)$

Find the equation of the tangent plane for the following indicated point.

(7) $xy + x^2z + 3xyz^2 = 0$ $(2,0,0)$

(8) $x^2 + xy^2 + xy + z^2 = 0$ $(1,1,\sqrt{3})$

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