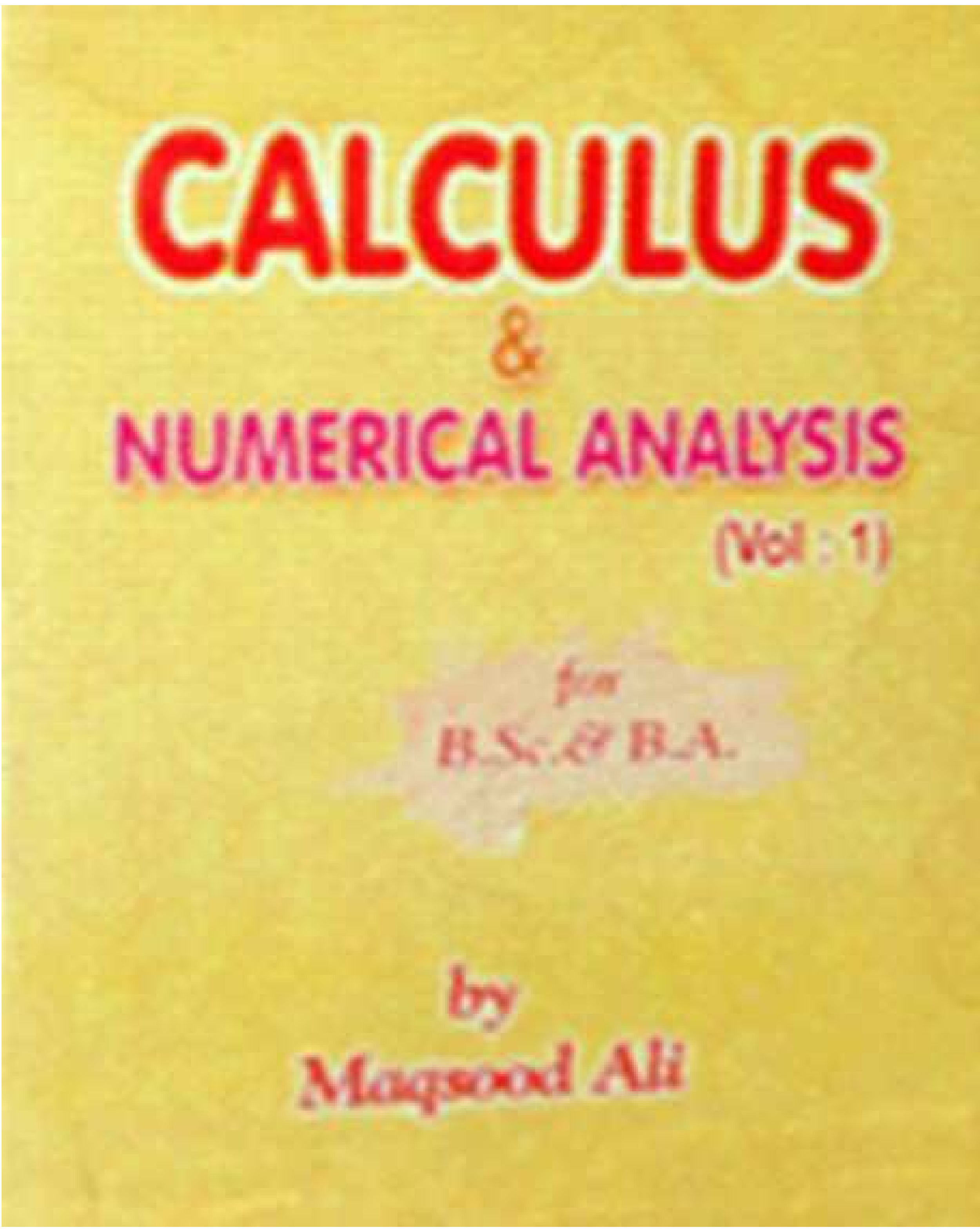
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
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CONTINUITY

Def-1: A function $f(x,y)$ is continuous at (a,b) if the following conditions are satisfied:

(i) $f(x,y)$ must be defined at $(x,y) = (a,b) \in D_f$

(ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ must exist

(iii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Def-2: A function $f(x,y)$ is continuous at $(a,b) \in D_f$ if for a given $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x,y) - f(a,b)| < \varepsilon$$

when $\quad |x-a| < \delta,\ |y-b| < \delta$

Figure C-15

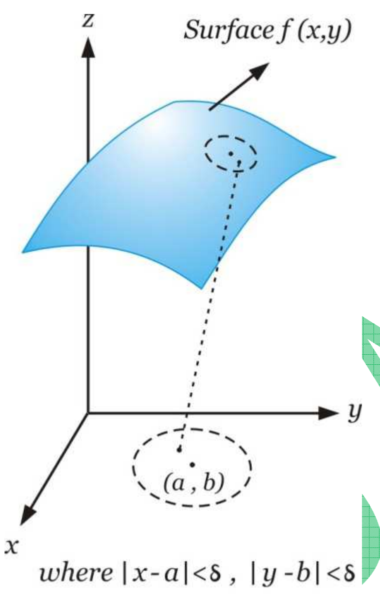

Figure C-15

Example 1:

Discuss the continuity for the following function

(i) $f(x,y) = \begin{cases} x^3 + 2y^2 + 3 & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

at $(0,0)$.

Solution:

(i) $$f(0,0) = 0$$

(ii) $$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} (x^3 + 2y^2 + 3) = 3$$

(iii) $$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) \neq f(0,0)$$

Hence f is discontinuous at $(0,0)$.

Example 2:

Discuss the continuity for the following function

$$f(x,y) = \begin{cases} x^2 - 3y & \text{for } (x,y) \neq (3,2) \\ 3 & \text{for } (x,y) = (3,2) \end{cases}$$

at $(3,2)$.

Solution:

(i) $f(3,2) = 3$

(ii)

$$\lim_{(x,y)\to(3,2)} f(x,y) = \lim_{(x,y)\to(3,2)} (x^2 - 3y)$$

$$= 9 - 6 = 3$$

(iii) $\lim_{(x,y)\to(3,2)} f(x,y) = f(3,2)$

f is continuous at(3,2).

Example 3:

Discuss the continuity for the following function

$$f(x,y) = \frac{x^2 + y^2}{x^2 - y^2}$$

$(x,y) \neq (0,0)$, $f(0,0) = 0$ at $(0,0)$

Solution:

(i) $f(0,0) = 0$

(ii)

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} \lim_{y\to 0} \frac{x^2 + y^2}{x^2 - y^2}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} \lim_{x\to 0} \frac{x^2 + y^2}{x^2 - y^2}$$

$$= \lim_{y\to 0} \frac{y^2}{-y^2} = -1$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exit.
Hence f is discontinuous at $(0,0)$.

Another Method:

$$f(x,y) = \frac{x^2+y^2}{x^2-y^2}$$

Suppose that for $\varepsilon>0$

$$x = \varepsilon\cos\theta \quad , \quad y = \varepsilon\sin\theta$$

$$\begin{aligned}|f(x,y)-f(0,0)| &= \left|\frac{x^2+y^2}{x^2-y^2} - 0\right| \\ &= \left|\frac{x^2+y^2}{x^2-y^2}\right| \\ &= \left|\frac{\varepsilon^2\cos^2\theta + \varepsilon^2\sin^2\theta}{\varepsilon^2\cos^2\theta - \varepsilon^2\sin^2\theta}\right| \\ &= \left|\frac{\varepsilon^2}{\varepsilon^2(\cos^2\theta - \sin^2\theta)}\right| \\ &= \left|\frac{1}{\cos 2\theta}\right| \not< \varepsilon\end{aligned}$$

Hence $f(x,y)$ is discontinuous at (0,0)

Example 2:

Discuss the continuity for the following function
$f(x,y) = \frac{xy}{x^2+y^2}$ at $(0,0)$ such that $f(0,0) = 0$.

Solution:

Suppose that
$x = \varepsilon\cos\theta$ and $y = \varepsilon\sin\theta$

$$\begin{aligned}|f(x,y)-f(0,0)| &= \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| \\ &= \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \\ &= \left|\frac{\varepsilon\cos\theta.\varepsilon\sin\varepsilon\theta}{\sqrt{\varepsilon^2\cos^2\theta + \varepsilon^2\sin^2\theta}}\right| \\ &= \left|\frac{\varepsilon^2\cos\theta\sin\theta}{\varepsilon^2\sqrt{\cos^2\theta+\sin^2\theta}}\right| \\ &= |\varepsilon\cos\theta\sin\theta| \\ &= \varepsilon.|\cos\theta|.|\sin\theta| < \varepsilon\end{aligned}$$

because $|\cos\theta| \leq 1$ and $|\sin\theta| \leq 1$
Hence $f(x,y)$ is continuous at $(0,0)$.

Example 3:

Discuss the continuity of the following functions at (0,0).

$$f(x,y) = \frac{x^4y^4}{(x^4+y^2)^3}$$

$$(x,y) \neq (0,0) \; ; \; f(0,0) = 0$$

Solution:

$$\lim_{x\to0}\lim_{y\to0} f(x,y) = \lim_{x\to0}\lim_{y\to0}\frac{x^4y^4}{(x^4+y^2)^3}$$

$$= \lim_{y\to0}\frac{0}{y^6} = 0$$

$$\lim_{y\to0}\lim_{x\to0} f(x,y) = \lim_{y\to0}\lim_{x\to0}\frac{x^4y^4}{(x^4+y^2)^3}$$

$$= \lim_{y\to0}\frac{0}{x^8} = 0$$

Let $y = g_1(x) = mx$

$$\lim_{x\to0} f(x, g_1(x)) = \lim_{x\to0}\frac{m^4x^8}{(x^4+m^2x^2)^3}$$

$$= \lim_{x\to0}\frac{m^4x^2}{(x^2m^2)^3} = 0$$

Let $y = g_2(x) = x^2$

$$\lim_{x\to0} f(x, g_2(x)) = \lim_{x\to0}\frac{x^4.x^8}{(x^4+x^4)^3}$$

$$= \lim_{x\to0}\frac{x^{12}}{8x^{12}}$$

$$= 1/8 \neq 0$$

Since $\lim_{x\to0} f(x, g_1(x)) \neq \lim_{x\to0} f(x, g_2(x))$

Hence f is discontinuous at (0,0).

Example 4:

Discuss the continuity of the following function at (2,3).

$$f(x,y) = \frac{(x-2)^2+(y-3)^2}{[(x-2)-(y-3)]^2}$$

$(x,y) \neq (2,3)$; $f(2,3) = 0$

Solution:

$$\lim_{x\to2}\lim_{y\to3} f(x,y) = 1 = \lim_{y\to3}\lim_{x\to2} f(x,y)$$

Let $y - 3 = m(x-2)$ or $y = m(x-2)+3$

$$\lim_{x\to2} f(x, g(x)) = \lim_{x\to2}\frac{(x-2)^2+m^2(x-2)^2}{[(x-2)-m(x-2)]^2}$$

$$= \frac{1+m^2}{(1-m)^2}$$

It depends on m.

Hence f is discontinuous at (2,3).

Example 5:

Discuss the continuity of the following function

$f(x,y) = |x| + |y|$ at (a,b).

Solution:

$$|f(x,y) - f(a,b)| = |(|x| + |y|) - (|a| + |b|)|$$
$$= (|x| - |a|) + (|y| - |b|)$$
$$\leq |(|x| - |a|)| + |(|y| - |b|)|$$
$$\leq |x - a| + |y - b|$$

If $|x - a| < \delta = \varepsilon/2$ and $|y - b| < \delta = \varepsilon/2$, then

$$|f(x,y) - f(a,b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$|f(x,y) - f(a,b)| < \varepsilon$$

Hence $f\ (x,y)$ is continuous at (a,b)

Theorem C-2:

If two functions $f(x,y)$ and $g(x,y)$ are both continuous at (a,b), so are the sum and product functions $f(x,y) + g(x,y)$ and $f(x,y)g(x,y)$ and so also is the quotient function $f(x,y)\ /\ g(x,y)$ provided that $g\ (a,b) \neq 0$.

Proof:

Since $f(x,y)$ and $g(x,y)$ are both continuous at (a,b).

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

and

$$\lim_{(x,y)\to(a,b)} g(x,y) = g(a,b)$$

(i) Let $\emptyset[f(x,y), g(x,y)] = f(x,y) + g(x,y)$

$$\lim_{\substack{x\to a\\ y\to b}} \emptyset[f(x,y), g(x,y)] = \lim_{\substack{x\to a\\ y\to b}}[f(x,y) + g(x,y)]$$

$$= \lim_{\substack{x\to a\\ y\to b}} f(x,y) + \lim_{\substack{x\to a\\ y\to b}} g(x,y)$$
$$= f(a,b) + g(a,b)$$
$$= \emptyset\ [f(a,b), g(a,b)]$$

So $\emptyset = f + g$ is continuous at (a,b).

Similarly we can prove that

$\emptyset = fg$ and $\emptyset = f/g$ are continuous at (a,b).

EXERCISE C-2

Show that the following functions are discontinuous at $(0,0)$, $(x, y) \neq (0,0)$ and $f(0,0) = 0$

(1) $f(x, y) = \dfrac{x^3 + y^3}{x^3 - y^3}$

(2) $f(x, y) = \dfrac{x^2 + xy}{xy + y^2}$

(3) $f(x, y) = \dfrac{x^2 + xy}{xy + y^2}$

(4) $f(x, y) = \dfrac{x^2 y^2}{x^3 + y^6}$

Discuss the continuity of the following functions at the indicated points.

(5) $f(x, y) = \begin{cases} x + y & \text{for} \quad (x, y) \neq (1,2) \\ x & \text{for} \quad (x, y) = (1,2) \end{cases}$
at $(1,2)$.

(6) $f(x, y) = \begin{cases} x^3 + y^2 & \text{for} \quad (x, y) \neq (2,2) \\ 5 & \text{for} \quad (x, y) = (2,2) \end{cases}$
at $(2,2)$.

(7) $f(x, y) = \dfrac{[(x - 2) - (y - 1)]^2}{(x - 2)^2 + (y - 1)^4}$ at $(2,1)$

Show that the following functions are discontinuous at the indicated points.

(8) $f(x, y) = \dfrac{(x - 1) - (y - 2)}{(x - 1) + (y - 2)}$ at $(1,2)$
such that $(x, y) \neq (1,2)$ and $f(1,2) = 0$

(9) $f(x, y) = \dfrac{(x - 4)^2 + (y - 5)^4}{[(x - 4) + (y - 5)]^4}$ at $(4,5)$
such that $(x, y) \neq (4,5)$ and $f(4,5) = 0$

(10) Show that $f(x, y) = sinx + cosy$ is continuous at (a, b).

(11) Show that $f(x, y) = cosx - siny$ is continuous at (a, b).

Show that the following functions are continuous at (0,0) where $f(0,0) = 0$.

(12) $f(x, y) = |x| + |y|$

(13) $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

(14) $f(x, y) = x^2 + xy + y^2$

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