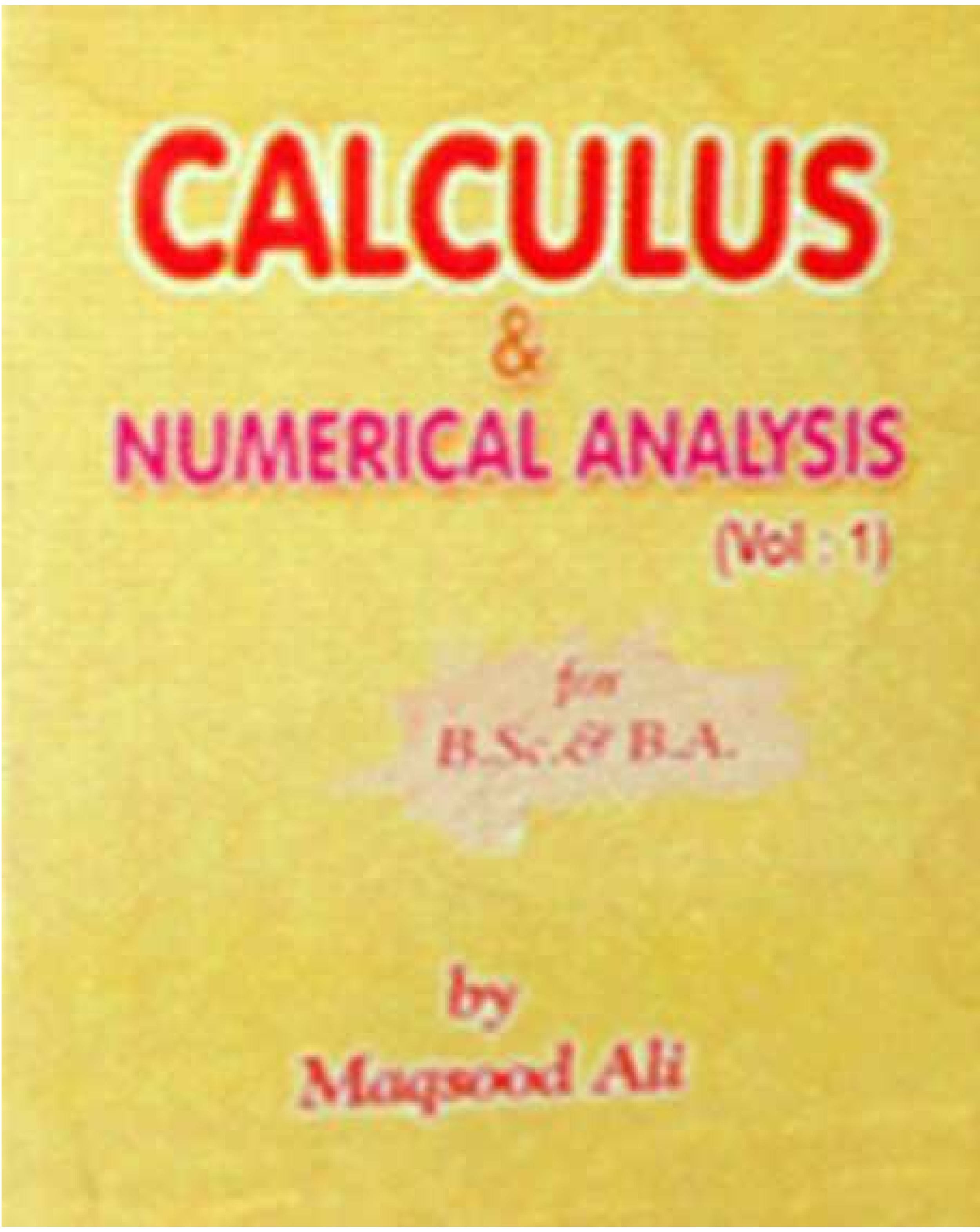
CALCULUS
&
NUMERICAL ANALYSIS
(Vol : 1)
for
B.Sc. & B.A.
by
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LIMITS

(i) Let $f(x, y)$ be a function of two variables in a domain D $\subset R^2$ and let (a, b) be any limit point of D. We say that "l" is the limit of $f(x, y)$ as x approaches "a" and y approaches "b" or (x, y) approaches (a, b).
It can be written as

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l \quad \text{or} \quad \lim_{(x,y)\to(a,b)} f(x, y) = l$$

(ii) For any positive number $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - l| < \varepsilon$$

whenever

$$\left||(x, y) - (a. b)|\right| < \delta$$

or

$$|x - a| < \delta \ and \ |y - b| < \delta$$

SIMULTANEOUS LIMIT:

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) \quad \text{or} \quad \lim_{(x,y)\to(a,b)} f(x, y)$$

is called simultaneous limit.

Repeated Limit or Iterated Limits:

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{x \to a}\{\lim_{y \to b} f(x, y)\}$$

and

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \lim_{y \to b}\{\lim_{x \to a} f(x, y)\}$$

are called repeated limits.
The repeated limits

$$\lim_{x \to a} \lim_{y \to b} f(x, y) \ and \ \lim_{y \to b} \lim_{x \to a} f(x, y)$$

are not necessarily equal.
Although they must be equal if

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

is to exist.
But the equality of repeated limits does not guarantee the existence of simultaneous limit.

Theorem C-1:
If

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

exist then

$$\lim_{x\to a}\lim_{y\to b} f(x,y) \text{ and } \lim_{y\to b}\lim_{x\to a} f(x,y)$$

must be equal
But

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)$$

does not guarantee the existence of

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

NON-EXISTENCE OF LIMIT:

(i) Repeated limit test:

Simultaneous limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does not exist if the repeated limits
$\lim_{x\to a}\lim_{y\to b} f(x,y)$ and $\lim_{y\to b}\lim_{x\to a} f(x,y)$ are not equal
.
that is

$$\lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y)$$

(ii) Two path Test:

Suppose that $g_1(x)$ and $g_2(x)$ are two functions such that

$$\lim_{x\to a} g_1(x) = b \; and \; \lim_{x\to b} g_2(x) = a$$

$f(x,y)$ is a function of two variables. The simultaneous limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does not exist , if

$$\lim_{x\to a} f(x, g_1(x)) \neq \lim_{x\to a} f(x, g_2(x))$$

Two-path test is shown in **figure C-7.**

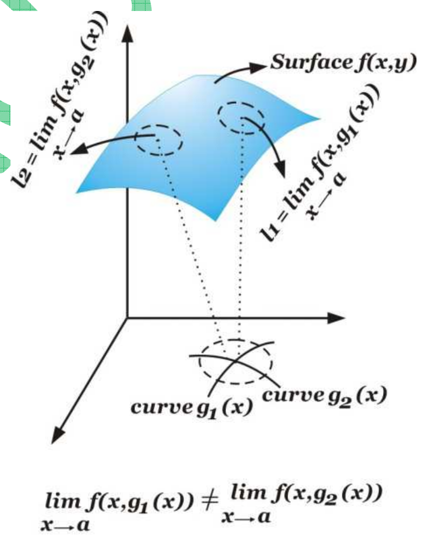

Figure C-7

Note: $y = g(x)$ or $x = g(y)$ is a straight line or a curve passes through the point (a, b) when $y = g(a) = b$ or $x = g(b) = a$ respectively.

$y = g(x)$ is a straight line or curve nearly passes through the point (a, b) or approaches to (a, b) when

$$\lim_{x\to a} g(x) = b.$$

Similarly $x = g(y)$ approaches to (a, b) when

$$\lim_{x \to b} g(y) = a$$

Path: Any curve or straight line (i.e. $y = g(x)$ or $x = g(y)\}$ nearly passes through the point (a, b) or approaches to (a, b) is a path.

Example 1:

Discuss the limit of the following function at (0,0)

$$f(x,y) = \frac{x - y^2}{\sqrt{x^2 + y^4}}$$

Solution:

$$\lim_{x\to 0}\lim_{y\to o} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{x - y^2}{\sqrt{x^2 + y^4}} = \lim_{x\to 0}\frac{x}{\sqrt{x^2}} = 1$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0}\frac{x - y^2}{\sqrt{x^2 + y^4}} = \lim_{y\to 0}\frac{-y^2}{y^2} = -1$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

So $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 2 :

Whether simultaneous limit exists or not for the following function at (0,0).

$$f(x,y) = \frac{x + y}{\sqrt{x^2 + y^2}}$$

Solution:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to o} f(x,y) = 1$$

Now we consider a function $y = g(x) = mx$, which is a straight line passes through $(0, 0)$

or

$$\lim_{x\to 0} g(x) = 0$$

$$\lim_{x\to 0} f(x, g(x)) = \lim_{x\to 0}\frac{x + mx}{\sqrt{x^2 + m^2x^2}}$$

$$= \lim_{x\to 0}\frac{x(1+m)}{x\sqrt{1+m^2}}$$

$$= \frac{1+m}{\sqrt{1+m^2}}$$

it depends on m , so the limits are not same for different values of m.

so $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Explanation:

Limit of f does not exist if the remaining value depends on m.

Since $g(x) = mx$

We can consider two functions g_1 and g_2 taking two different values of m.

If $m = 1 \quad \Rightarrow \quad g_1(x) = x$

If $m = 100 \quad \Rightarrow \quad g_2(x) = 100x$

$$\lim_{x\to 0} f\big(x, g_1(x)\big) = \frac{1+1}{\sqrt{1+1^2}} = \sqrt{2} = 1.4$$

$$\lim_{x\to 0} f\big(x, g_2(x)\big) = \frac{1+100}{\sqrt{1+100^2}} = 0.01$$

$$\lim_{x\to 0} f(x_1, g_1(x)) \neq \lim_{x\to 0} f(x, g_2(x))$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist,

as shown in

figure C-8.

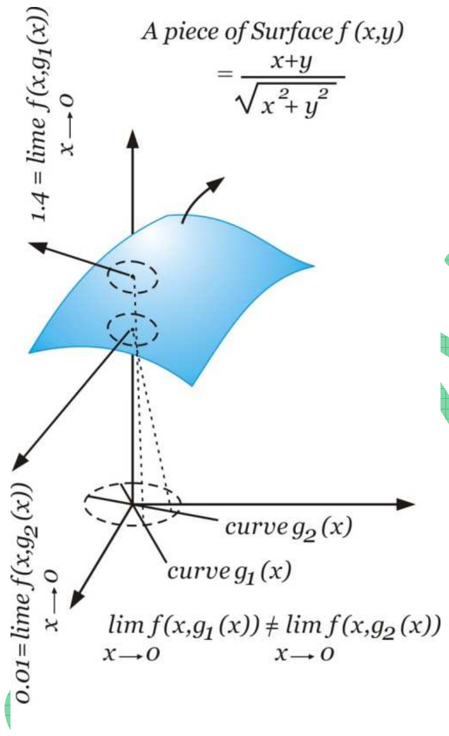

Figure C-8

Example 3:

Discuss the limit of the following function f at (3,5)

$$f(x,y) = \frac{(x-3)^2(y-5)^2}{(x-3)^6+(y-5)^3}$$

Solution:

$$\lim_{(x,y)\to(3,5)} f(x,y) = \lim_{(x,y)\to(3,5)} \frac{(x-3)^2(y-5)^2}{(x-3)^6+(y-5)^3}$$

Suppose that

$$y-5 = m\,(x-3)^2$$
$$g(x) = y = m\,(x-3)^2+5$$
$$\lim_{x\to 3} g(x) = 5$$

So that,

$$\lim_{x\to 3} f\big(x, g(x)\big) = \lim_{x\to 3} \frac{(x-3)^2\{m(x-3)^2\}^2}{(x-3)^6+\{m(x-3)^2\}^3}$$
$$= \lim_{x\to 3} \frac{m^2(x-3)^6}{(x-3)^6(1+m^3)}$$
$$= \frac{m^2}{1+m^3}$$

It depends on m, so limit of f does not exist at (3,5).

Explanation:

Suppose that g_1 and g_2 are the two functions for $m = 1$ and $m = -9/10$ respectively.

$g_1(x) = (x-3)^2 + 5$

$g_2(x) = -\frac{9}{10}(x-3)^2 + 5$

$$\lim_{x\to 3} f\big(x, g_1(x)\big) = 0.5$$

$$\lim_{x\to 3} f\big(x, g_2(x)\big) = 2.99$$

$$\lim_{x\to 3} f(x, g_1(x) \neq \lim_{x\to 3} f(x, g_2(x))$$

Hence limit of the function does not exist at (3,5).

Figure c-9

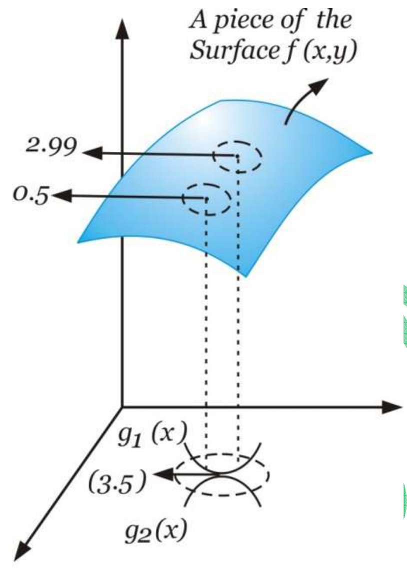

Figure C-9

Example 4:

Discuss $\lim_{(x,y)\to(0,0)} f(x,y)$ for the following functions.

(i) $f(x,y) = \dfrac{y^3 + 2xy - x^3}{y^3 + x^3}$

(ii) $f(x,y) = x^3 + xy^2$

Solution: (i)

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{y^3 + 2xy - x^3}{y^3 + x^3}$$

$$= \lim_{x\to 0}\frac{-x^3}{x^3} = -1$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0}\frac{y^3 + 2xy - x^3}{y^3 + x^3}$$

$$= \lim_{y\to 0}\frac{y^3}{y^3} = 1$$

Since $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(ii) $f(x,y) = x^3 + xy^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^3 + xy^2) = 0$$

Difference between Two-path Test and Repeated Limit:

Repeated limit is a particular case of two-path test because for repeated limit we discuss the limit of $f(x,y)$ along only two different paths $y \cong b$ and $x \cong a$ when $y \cong b$ is a straight line parallel to x-axis and $x \cong a$ is another straight line parallel to y-axis , as shown in **figure C-10.**

{**Note** : $x \cong a$ $mean$ $x = a^{+} or$ $x = a^{-}$}

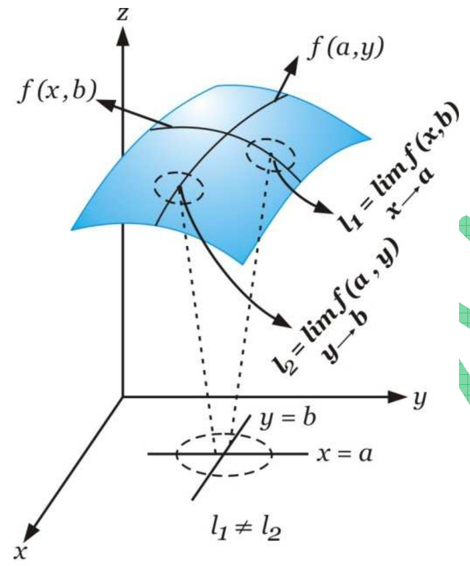

Figure C-10

Existence of limit:

(i) Let $z = f(x, y)$,where f is a function of two variables. The limit of f at the point (a, b)exists if the limit is same along every approach path as shown in the **figure C-11**.

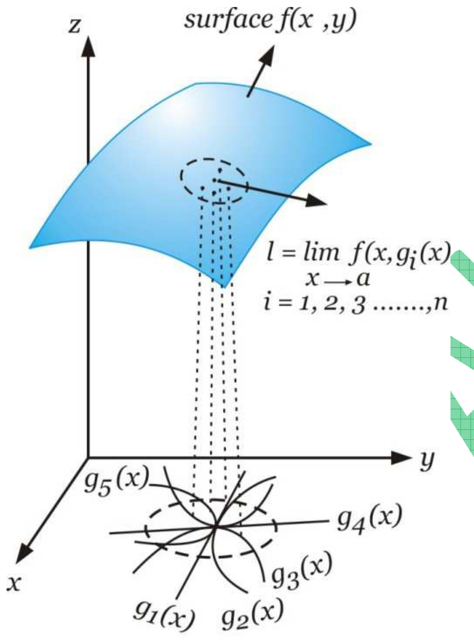

Figure C-11

$g_i(x)$ and $g_j(x)$ are any two paths through the deleted neighborhood of (a, b).
The limit of f exist when

$$\lim_{x \to a} f(x, g_i(x)) = \lim_{x \to a} f\left(x, g_j(x)\right)$$

$$i \neq j \qquad , \quad i = 1,2,3, \dots, n$$
$$j = 1,2,3, \dots, n$$

(ii) The limit of f exist, when

$$|f(x, y) - l| < \varepsilon$$

such that

$$|x - a| < \delta \quad , \quad |y, -b| < \delta$$

Figure C-12

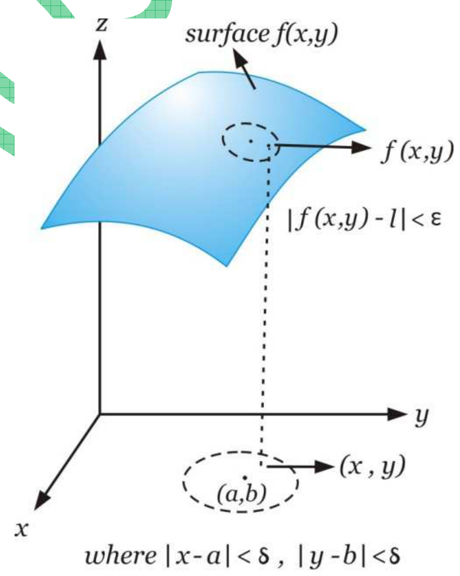

Figure C-12

Difference between the limit of one variable function and two variables function:

$Let\ y = f(x)$, where f is a function of one variable x. To discuss the limit of $f\ at\ x = a$, we find the limit for two values a^- and a^+lies in an open interval about a.The limits are called left hand limit and right hand limit respectively.

The $\lim\limits_{x \to a} f(x)$ exist if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$Let\ z = f(x, y)$, where f is a function of two variables. To discuss the limit of $f\ at\ (a, b)$, there are a lot of points lie in the open disk about (a, b) these points lie on the different curves or straight lines passes through the open disk as shown in the **figure C-14**. The limit exists only when all the limits are equal.

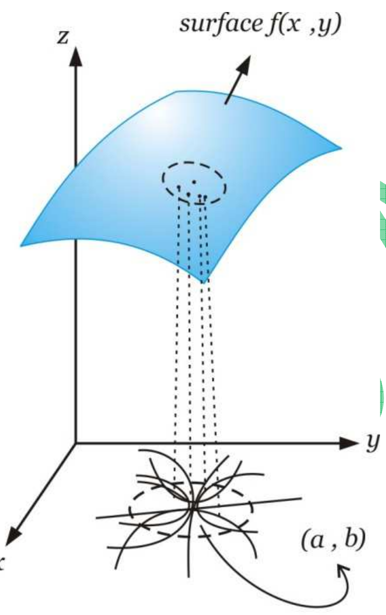

Figure C-14

EXERCISE C-1

Find the limit of the following functions at the indicated point:

(1) $f(x,y)=\dfrac{x^3-y^3}{x-y}$ at $(1,1)$

(2) $f(x,y)=\dfrac{xy}{x^2-xy}$ at $(0,3)$

(3) $f(x,y)=\dfrac{x^2-2xy+y^2}{x-y0}$ at $(2,2)$

(4) $f(x,y)=\frac{x+y}{x-y}$ at $(3,2)$

Show that the simultaneous limit does not exist at (0,0) for the following functions;

(5) $f(x,y)=\dfrac{x^2y^2}{x^6+y^3}$ (6) $f(x,y)=\dfrac{x^2y^2}{x^3+y^6}$

(7) $f(x,y)=\dfrac{x^3-y^3}{x^3+y^3}$ (8) $f(x,y)=\dfrac{x^2+xy}{xy+y^2}$

Discuss the limit of the following functions at the indicated point.

(9) $f(x,y)=\dfrac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$ at $(2,1)$

(10) $f(x,y)=\dfrac{(x-a)^2+(y-b)}{\sqrt{(x-a)^4+(y-b)^2}}$ at (a,b)

(11) $f(x,y)=\dfrac{(x-1)+(y-2)^2}{\sqrt{(x-1)^2+(y-2)^2}}$ at $(1,2)$

(12) $f(x,y)=\dfrac{x(y-1)}{x^2+(y-2)^2}$ at $(0,1)$

(13) $f(x,y)=\dfrac{(x-a)^2y^2}{\sqrt{(x-a)^6+y^{12}}}$ at $(a,0)$

Discuss the limit of the following functions along the given path at the indicated point.

(14) $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ at (0,0)

along the paths

(i) $\boldsymbol{y=x}$

(ii) $\boldsymbol{x=my}$

(iii) $\boldsymbol{y=3x}$

(iv) $\boldsymbol{x=5y}$

(15) $f(x,y)=\dfrac{x^3+y}{\sqrt{x^6+y^2}}$ at (0,0)

along the paths

(i) $y=2x^3$

(ii) $x=3y^{1\backslash 3}$

(iii) $y=3x^3$

(iv) $x=my^{1\backslash 3}$

(16) $f(x,y)=\dfrac{(x-2)^2+(y-3)}{\sqrt{(x-2)^4+(y-3)^2}}$ at (2,3)

along the paths

(i) $y-3=(x-2)^2$

(ii) $y-3=m(x-2)^2$

(iii) $x-2=m(y-3)^{1\backslash 2}$

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