BOOK 1

CALCULUS

WITH APPLICATIONS

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Chapter 49

SMALL INCREAMENT AND DIFFERENTIAL

DIFFERENTIAL:

Let $y = f(x)$ and f is differentiable at $x = x_o$. If differential dx is a quantity, which is the difference of the changes from x_o to $x_o + dx$, then differential dy is defined by

$$dy = f'(x)\, dx$$

SMALL INCREMENT

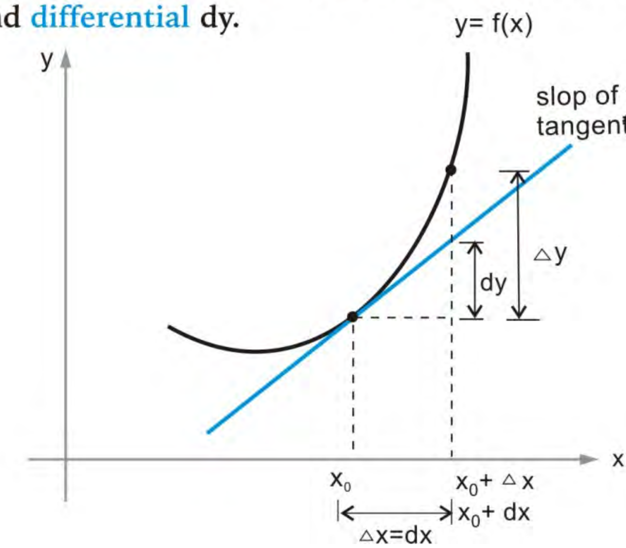
f is a function of x and $y = f(x)$. Suppose that x changes from x_o to $x_o + \Delta x$, there is a corresponding change in y from $f(x_o)$ to $f(x_o + \Delta x)$. If Δx is the small increment in x and Δy is the corresponding increment in y then

$$\Delta x = (x_o + \Delta x) - x_o$$

and $$\Delta y = f(x_o + \Delta x) - f(x_o)$$

SMALL INCREAMENT AND DIFFERENTIAL

Consider the following graph of a function to understand the difference between increment Δy and differential dy.



Suppose that differential dx and increment Δx are equal. $f'(x_o)$ is the slope of the tangent at $x = x_o$. The slope of the tangent $f'(x_o)$ at point x_o is the ratio of the differential dx and corresponding differential dy at x_o, as shown in the figure.

$$\left(\frac{dy}{dx}\right)_{x=x_o} = f'(x_o)$$

The differential dy is written as

$$dy = f'(x_o)\, dx$$

The tangent line is the graph of a linear function. So we can say that Δy is the change in the value of the function of the curve and dy is the change of the value of corresponding linear function (tangent line).

So that Δy is the true change and dy is approximate change in f(x).

DIFFERENCE BETWEEN Δy AND dy

Since

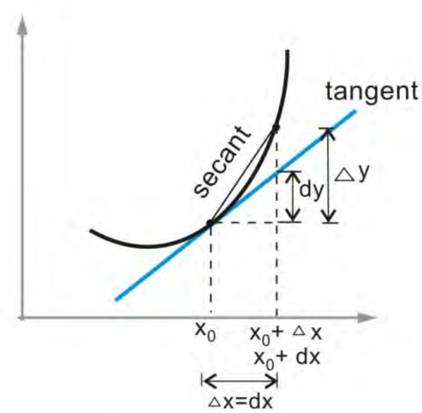
$$\Delta y = f(x_o + \Delta x) - f(x_o)$$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

$$\Delta y = \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} \cdot \Delta x$$

$$\Delta y = (\text{slope of the secant}) \cdot \Delta x \longrightarrow (1)$$



As we have defined that the gradient is

$$dy = f'(x_o) \cdot dx$$

$$\therefore \quad dy = (\text{slope of the tangent at } x_o) \cdot dx \longrightarrow (2)$$

By comparing (1) and (2), if $\Delta x = dx$,

"The only difference in increment Δy and differential dy is that for dy we use slope of the tangent instead of slope of the secant."

Δy nearly Equal to dy:

If Δx is very small, then secant is almost parallel to the tangent. So slopes of secant and tangent are equal. Thus

The differential dy:

$$\begin{aligned} dy &= f'(4)\ .\ dx \\ &= 2(4)\ .\ (4.01 - 4) \\ &= 0.08 \end{aligned}$$

Δy is nearly equal to dy. Both the values are exactly equal to three decimal places.

Example 2: Find the approximate value of sin 31 °.

Solution:-

$$\text{Sin } 31^\circ = \text{Sin}\,(30^\circ + 1^\circ) = \sin\,(x + \Delta x)$$

where $x = 30^\circ$

and

$$\Delta x = 1^\circ = 1 \times \frac{\pi}{180} \text{rads.} = 0.01745 \text{ (correct to 4 significant figures)}$$

Suppose that sin x = y.

Let Δx be the increment in x and Δy is the corresponding increment in y

$$\begin{aligned} \sin\,(x + \Delta x) &= y + \Delta y \\ &\approx y + \frac{dy}{dx}\,.\,\Delta x \\ &= \sin x + \frac{d}{dx}(\sin x)\,.\,\Delta x \\ &= \sin x + \cos x\,.\,\Delta x \end{aligned}$$

Substitutte the values of x and Δx.

$$\sin\,(30^\circ + 1^\circ) \approx \sin 30^\circ + \cos 30^\circ\ (0.01745)$$

$$\sin 31^\circ \approx 0.515$$

Example 3: The radius of circle increase from 5 cm to 5.03 cm. Determine the approximate change in the area of the circle.

Solution:-

increment in radius = Δr = 5.03 – 5 = 0.03 cm

increment in area = ΔA = ?

The area of a circle with radius r is

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

Since Δr is very small, so

$$\Delta A \approx \frac{dA}{dr} . \Delta r$$
$$= 2\pi r . \Delta r$$

Substitute r = 5 cm and Δr = 0.03 cm

$$\Delta A \approx 2\pi (5)(0.03)$$
$$= 0.942 \text{ cm}^2 \text{ (correct to three significan t figures)}$$

Example 4: The area of a circle changes from 50 cm^2 to 50.08 cm^2. Find the change in the radius of the circle.

Solution:-

change in area = ΔA = 50.08 – 50 = 0.08 cm^2

change in radius = Δr = ?

The area of the circle with radius r is

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

For the value of r

$$A = \pi r^2$$
$$50 = \pi r^2$$
$$r = 3.99 \text{ cm (corret to two decimal places)}$$

Since ΔA is very small, so

$$\Delta r \approx \frac{dr}{dA} . \Delta A$$

or
$$= \frac{1}{\frac{dA}{dr}} . \Delta A$$
$$= \frac{1}{2\pi r} . \Delta A$$

r = 3.99 cm and ΔA = 0.08 cm^2

$$\Delta r \approx \frac{1}{2\pi (3.99)} (0.08)$$
$$= 0.0032 \text{ cm (correct to four decimal places)}$$

EXERCISE K-14

(1) Given that $\sin 30^\circ = 0.5$. Calculate the approximate value of $\sin 31^\circ$.

(2) Given that $\tan 45^\circ = 1$. Calculate the approximate value of $\tan 44^\circ$.

(3) Given that $\cos 90^\circ = 0$. Calculate the approximate value of $\cos 89^\circ$.

(4) Given that $\ln 6 = 1.79$. Calculate the approximate value of $\ln 6.1$.

(5) Given that $\log_{10} 8 = 0.903$. Calculate the approximate value of $\log_{10} 7.9$.

(6) The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Find the approximate change in V when radius changes from 3 to 3.01.

(7) The area of a circle is $A = \pi r^2$. Find the approximate increased in A when radius increases from 5 to 5.01.

(8) The volume of the cube is $V = x^3$ of side x. Find the approximate decreased in x if volume of the cube decreases from 27 to 26.7.